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Aggregation Failure in Stochastic OLG Economies with Production

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Abstract

We study whether aggregate macroeconomic dynamics in stochastic overlapping generations economies with production can be summarized by a low-dimensional state. In a three-period OLG economy with state-contingent returns, we show that aggregation fails generically: even when aggregate capital arises endogenously, it does not summarize the payoff-relevant distributional information required for asset pricing. We derive cross-cohort Euler compatibility restrictions implied by such representations and show that they are generically violated. Consequently, no low-dimensional aggregate-state representation is consistent with equilibrium; equilibrium must be recursive with additional endogenous states. This implies sequential market incompleteness and potential inefficiency. Benchmark cases clarify that aggregation fails due to multi-period lifetimes, state-contingent returns, and post-shock portfolio rebalancing.

Keywords: Overlapping generations, Aggregate uncertainty, Aggregation, State representation, Recursive equilibrium, Sequential market incompleteness
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1. Introduction

A central question in macroeconomic dynamics is whether equilibrium prices and allocations can be represented as functions of a low-dimensional aggregate state. Many recursive macroeconomic models rely on this principle, treating variables such as capital and the current shock as sufficient to summarize the information relevant for forecasting and pricing. This paper studies when such a representation is valid in stochastic overlapping-generations (OLG) economies with production and capital accumulation.

The main result shows that this aggregation property fails generically. Even though capital arises as a natural endogenous state variable in production economies, it does not, in general, summarize the payoff-relevant distributional information required for asset pricing across cohorts. The reason is that multi-period lifetimes, state-contingent returns, and post-shock portfolio rebalancing generate cohort-specific wealth effects that are not captured by aggregate capital alone. As a result, a representation based only on capital and the current shock is generically inconsistent with equilibrium.

This contrasts with exchange economies, where the absence of a sufficient aggregate state variable is immediate. In the present setting, a natural aggregate state—capital—emerges endogenously, yet aggregation still fails. The paper thus identifies a stronger breakdown of aggregate-state representations: even when an endogenous aggregate is available, it is generically insufficient to support a low-dimensional description of macroeconomic dynamics.

Formally, the analysis derives cross-cohort Euler compatibility restrictions implied by a state representation based only on aggregate capital and the current shock, and shows—via a residual-set argument—that these restrictions are generically violated. Consequently, no low-dimensional aggregate-state representation is consistent with equilibrium. The relevant equilibrium concept must therefore be recursive, requiring additional endogenous state variables. This implies that sequential markets are generically incomplete and that equilibrium allocations need not be short-run interim Pareto efficient.

These results have direct implications for macroeconomic modeling. In environments with long-lived agents and aggregate risk, distributional information generated by sequential trading remains payoff-relevant and cannot, in general, be summarized by aggregate variables alone. The analysis therefore highlights a fundamental limitation of low-dimensional state representations in macroeconomic models with overlapping generations and production.

The remainder of the paper is organized as follows. Section 2 reviews related work. Section 3 presents the environment and equilibrium notions. Section 4 states the main result and derives the compatibility restrictions. Section 5 develops the recursive equilibrium representation and establishes endogenous sequential market incompleteness. Section 6 discusses welfare implications. Finally, Section 7 provides additional intuition for the failure of aggregation in the production economy.

2. Related literature

This paper is related to several strands of the literature on stochastic overlapping-generations (OLG) economies, aggregation and state representation, and market incompleteness.

A first strand studies equilibrium dynamics and memory in stochastic OLG models. Early work by Spear (1985) shows that even with i.i.d. aggregate shocks, rational expectations equilibria in OLG economies may exhibit serial correlation in prices and allocations when more than one good is traded. Building on this insight, Citanna and Siconolfi (2007) show that short-memory equilibria—the stochastic analogue of steady states or low-order cycles in deterministic OLG models—fail to exist generically in exchange economies. These results highlight that demographic heterogeneity alone can generate endogenous state dependence in equilibrium.

A second strand focuses on recursive equilibrium and the dimensionality of the equilibrium state space. Duffie et al. (1994) establish general existence results for stationary Markov equilibria in stochastic economies, including OLG environments, but allow the state space to include potentially many endogenous variables. More recently, Citanna and Siconolfi (2010) prove the generic existence of recursive equilibrium in stochastic OLG economies with sufficient intragenerational heterogeneity, clarifying the conditions under which equilibria can be represented as time-homogeneous Markov processes. The present paper complements this work by showing that, even when recursive equilibria exist, restricting attention to a minimal (short-memory) state space generically leads to nonexistence.

The paper is also related to work on stochastic OLG economies with production and minimal-state-space recursive representations. In two-period environments, or more generally in settings where no cohort makes portfolio decisions after the realization of uncertainty, aggregate capital can be sufficient to summarize equilibrium dynamics. This perspective is reflected in, among others, Barbie et al. (2007) and McGovern et al. (2012). The paper here demonstrates that once agents live at least three periods and continue to trade after the realization of aggregate shocks, this aggregation property breaks down generically because state-contingent returns generate cross-cohort pricing restrictions that cannot be captured by aggregate capital alone.

Closest to this paper is Henriksen and Spear (2012), who show that in stochastic OLG economies with productive assets equilibrium prices generally cannot be represented as functions of a low-dimensional aggregate state. They interpret this non-Markovian dependence as endogenous market incompleteness and relate the resulting recursive equilibria to sequential market incompleteness and failures of short-run interim efficiency. The present paper adopts this equilibrium-memory perspective but asks a distinct question in a production setting with capital accumulation: whether the endogenous law of motion of capital can restore a short-memory representation. The answer is negative. The paper shows that even when aggregate capital is present as a natural endogenous state variable, it does not generically summarize the payoff-relevant distributional information created by post-shock portfolio rebalancing.

Orrego and Spear (2011) examine a stochastic three-period OLG economy with production in which agents trade a riskless bond and risky capital. They show that a stationary equilibrium in which prices and allocations depend only on aggregate capital and the contemporaneous shock does not exist, so that a recursive formulation with additional endogenous state variables is required. The present paper takes this production-economy observation as a starting point and provides a sharper analytical characterization: it derives explicit cross-cohort Euler compatibility restrictions implied by short memory and shows—via a residual-set argument—that these restrictions are generically violated. In this sense, the contribution of the paper is to turn a model-specific nonexistence result into a general equilibrium-memory theorem for stochastic OLG production economies.

Finally, the welfare analysis in this paper is related to the literature on sequential completeness and interim efficiency in economies with finitely lived agents and sequential markets. Demange (2002) provides a general characterization of short-run interim efficiency; consistent with this framework, the paper shows that the recursive equilibria implied by the failure of short memory are generically sequentially incomplete and therefore need not be short-run interim Pareto efficient.

3. Environment and equilibrium notions

Time is discrete and infinite, $t = 0, 1, 2, \ldots$. In each period a new cohort is born and lives for three periods (young, middle-aged, retired). There is no population growth.

3.1. Model primitives

Aggregate uncertainty. Aggregate uncertainty is generated by a finite-state exogenous shock process $\{s_t\}$ with $s_t \in \{h, l\}$, where h and l denote high and low realizations. For concreteness, the baseline assumes $\{s_t\}$ is i.i.d. with $\Pr(s_t = h) = \pi_h \in (0, 1)$ and $\Pr(s_t = l) = \pi_l = 1 - \pi_h$.

Technology and firms. At date t a representative firm produces the consumption good using predetermined capital K_{t-1} (chosen at date $t-1$) and contemporaneous labor L_t after observing the aggregate shock $s_t \in \{h, l\}$:

$$Y_t = \eta(s_t) F(K_{t-1}, L_t).$$

Capital depreciates at rate $\delta \in [0, 1]$, so undepreciated capital carried into date t contributes $(1-\delta)K_{t-1}$ at the end of date t. Under competitive factor pricing (normalizing labor to $L_t = 1$), wages and the gross return on capital carried from $t-1$ into t are

$$w_t \equiv w(K_{t-1}, s_t) = \eta(s_t) F_L(K_{t-1}, 1), \qquad r_t \equiv r(K_{t-1}, s_t) = \eta(s_t) F_K(K_{t-1}, 1) + (1-\delta),$$

where $\eta(s) > 0$ is a productivity multiplier with $\eta(h) > \eta(l)$, and F is constant returns to scale, C^2, strictly increasing and strictly concave in K on the relevant domain. The strict inequality $\eta(h) > \eta(l)$ imposes nondegenerate aggregate risk; if $\eta(h) = \eta(l)$, aggregate uncertainty becomes payoff-irrelevant and the state-contingent return mechanism underlying Proposition 1 collapses.

Households and preferences. Preferences are time-separable with expected utility

$$\mathbb{E}\big[u(c_y) + \beta u(c_m) + \beta^2 u(c_r)\big],$$

where $u : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ is continuous on $[0, \infty)$, C^2 on $(0, \infty)$, strictly increasing, strictly concave, and satisfies Inada conditions on $(0, \infty)$, and $\beta \in (0, 1]$.

Labor endowments. Agents supply labor inelastically when young and middle-aged. Let $\ell_y, \ell_m \geq 0$ with $\ell_y + \ell_m = 1$. Labor income is $w_t \ell_y$ when young, $w_{t+1}\ell_m$ when middle-aged, and zero in retirement.

Assets and timing. Agents trade two one-period securities each period: (i) a riskless bond b in zero net supply purchased at price $q_t = q(K_{t-1}, s_t)$ that pays one unit at date $t+1$ in every state; and (ii) physical capital a purchased at date t that yields the gross return $r_{t+1} = r(K_t, s_{t+1})$ at date $t+1$. At the beginning of period t, the shock s_t is realized and factor prices (w_t, r_t) are determined as functions of the current state (K_{t-1}, s_t). Young and middle-aged agents then choose consumption and portfolios.

Budget constraints and admissibility. With young positions $(a_{y,t}, b_{y,t})$ chosen at t and middle-aged positions $(a_{m,t+1}, b_{m,t+1})$ chosen at $t+1$, the period budget constraints are

$$\begin{aligned} c_{y,t} &\leq w_t \ell_y - q_t b_{y,t} - a_{y,t}, \\ c_{m,t+1} &\leq w_{t+1}\ell_m + b_{y,t} + r_{t+1}a_{y,t} - q_{t+1}b_{m,t+1} - a_{m,t+1}, \\ c_{r,t+2} &\leq b_{m,t+1} + r_{t+2}a_{m,t+1}, \end{aligned}$$

with $c_{i,t} \geq 0$ for $i \in \{y, m, r\}$. Here $w_t = w(K_{t-1}, s_t)$ and $q_t = q(K_{t-1}, s_t)$ are functions of the current aggregate state (K_{t-1}, s_t), while the payoff on capital purchased at t is $r_{t+1} = r(K_t, s_{t+1})$ with $K_t = K^+(K_{t-1}, s_t)$. Natural debt limits are assumed to be chosen so that, together with non-negativity of consumption and bounded prices/returns on the relevant equilibrium range, each household's feasible set is nonempty, closed, and bounded. These admissible portfolio sets ensure that the individual optimization problems below are well posed.

Individual optimization problems. At date t, a young agent with current state (K_{t-1}, s_t) solves

$$\max_{c_y, a_y, b_y} \; u(c_y) + \beta\, \mathbb{E}_t\Big[V_m\big(K_t, s_{t+1}; a_y, b_y\big)\Big]$$

subject to the young budget constraint and the admissible portfolio set, where $K_t = K^+(K_{t-1}, s_t)$ and V_m denotes the continuation value when middle-aged. At date $t+1$, a middle-aged agent with inherited young portfolio $(a_{y,t}, b_{y,t})$ and current state (K_t, s_{t+1}) solves

$$\max_{c_m, a_m, b_m} \; u(c_m) + \beta\, \mathbb{E}_{t+1}\Big[u\big(c_r(K_{t+1}, s_{t+2})\big)\Big]$$

subject to the middle-aged and retirement budget constraints and the admissible portfolio set. These explicit problems clarify that young and middle-aged agents both solve forward-looking portfolio problems, but only the latter chooses after observing the current shock realization.

Notation (state dependence). State dependence is always written in parentheses. The current aggregate state at date t is (K_{t-1}, s_t): capital is predetermined and equals K_{t-1} when the shock s_t is realized. Accordingly,

$$q_t = q(K_{t-1}, s_t), \qquad r_t = r(K_{t-1}, s_t), \qquad w_t = w(K_{t-1}, s_t).$$

Moreover, next period's predetermined capital is determined by current portfolio choices via the law of motion

$$K_t = K^+(K_{t-1}, s_t) \equiv a_y(K_{t-1}, s_t) + a_m(K_{t-1}, s_t).$$

Household policy rules are written similarly and evaluated along the equilibrium path at (K_{t-1}, s_t); for example,

$$a_{y,t} = a_y(K_{t-1}, s_t), \qquad b_{y,t} = b_y(K_{t-1}, s_t), \qquad c_{y,t} = c_y(K_{t-1}, s_t).$$

When it is useful to track a life-cycle history tag (e.g. the shock realized when a cohort was young), an additional argument is added separated by a semicolon, e.g. $c_m(K, s; \sigma)$ for $\sigma \in \{h, l\}$. (Commas separate ordinary function arguments, while the semicolon is reserved exclusively for life-cycle history tags.) An equilibrium path is a sequence

$$\{(K_t, s_t, q_t, r_t, w_t, c_{y,t}, c_{m,t}, c_{r,t}, a_{y,t}, a_{m,t}, b_{y,t}, b_{m,t})\}_{t \geq 0}$$

such that, for a given initial condition (K_{-1}, s_0), the sequence is generated by the policy rules and satisfies the conditions of Definition 1 at every date.

Definition 1 (Short-memory (strongly stationary) competitive equilibrium)**.** *For the purposes of this paper,* short-memory *and* strongly stationary *are used interchangeably to mean that equilibrium prices and allocations are measurable with respect to the current aggregate state only. For every current aggregate state $(K_{t-1}, s_t) = (K, s)$, a short-memory competitive equilibrium consists of price functions $q(K, s)$, $r(K, s)$, and $w(K, s)$ and household decision rules $c_y(K, s)$, $a_y(K, s)$, $b_y(K, s)$, $c_m(K, s)$, $a_m(K, s)$, $b_m(K, s)$, and $c_r(K, s)$, defined for $s \in \{h, l\}$ and for K in the equilibrium range, such that:*

(i) ***Household optimality.*** *Given prices, young and middle-aged agents solve the optimization problems above.*

(ii) ***Firm optimality.*** *Prices satisfy competitive factor pricing based on predetermined capital K:*

$$w(K,s) = \eta(s)F_L(K,1), \qquad r(K,s) = \eta(s)F_K(K,1) + (1-\delta).$$

(iii) ***Market clearing.*** *$L = 1$, $b_y(K,s) + b_m(K,s) = 0$, and next period's predetermined capital is*

$$K^+(K,s) \equiv a_y(K,s) + a_m(K,s).$$

Along an equilibrium path, this means $K_t = K^+(K_{t-1}, s_t)$.

(iv) ***Short-memory restriction.*** *All equilibrium objects are measurable functions of the current aggregate state (K_{t-1}, s_t); in particular, prices and allocations at date t can be written as functions of (K_{t-1}, s_t) only.*

4. Impossibility of short-memory equilibrium

Throughout the analysis, a short-memory equilibrium corresponds to an aggregate-state representation in which prices and allocations depend only on (K_{t-1}, s_t). The impossibility result is driven by the interaction between the short-memory restriction and post-shock decision-making by multi-period agents. We therefore begin by deriving a direct implication of strong stationarity alone. Since middle-aged agents choose consumption after observing the current aggregate shock, short memory requires their consumption rule to depend only on the current aggregate state. The next lemma formalizes this implication by showing that, under strong stationarity, middle-aged consumption must be invariant to the shock realized when the cohort was young.

Lemma 1 (Short-memory implies history erasure at middle age)**.** *In any short-memory competitive equilibrium, the middle-aged consumption rule depends only on the current aggregate state. Hence, conditional on the state (K_t, s_{t+1}), middle-aged consumption is invariant to the young-history shock. Equivalently, for each $s' \in \{h, l\}$ and each K_t in the equilibrium range,*

$$c_m(K_t, s'; h) = c_m(K_t, s'; l).$$

This property is implied by Definition 1 and does not rely on preferences or returns.

Proof. This is a direct implication of the short-memory restriction in Definition 1: conditional on the current state, the middle-aged policy rule cannot depend on the young-history tag. A detailed proof is given in Appendix A.1. □

The next lemma shows how the history-erasure implication of short memory (Lemma 1) interacts with state-contingent returns.

Lemma 2 (State-independence of young portfolios under nondegenerate returns)**.** *Assume a short-memory equilibrium exists. Fix a cohort born at date t. Given the current aggregate state (K_{t-1}, s_t), let the cohort's young portfolio choice at date t be*

$$(a_{y,t}, b_{y,t}) = \big(a_y(K_{t-1}, s_t),\, b_y(K_{t-1}, s_t)\big).$$

Suppose that capital returns at date $t+1$ differ across shock realizations at the relevant predetermined capital level, i.e.

$$r(K_t, h) \neq r(K_t, l),$$

where K_t is next period's predetermined capital induced by market clearing at date t. Economically, this condition means that aggregate uncertainty is payoff-relevant for asset pricing: different realizations of the productivity shock imply different capital returns at the same predetermined capital stock. Then the young portfolio is independent of the young shock:

$$a_y(K_{t-1}, h) = a_y(K_{t-1}, l), \qquad b_y(K_{t-1}, h) = b_y(K_{t-1}, l).$$

Proof. By Lemma 1, middle-aged consumption is invariant to the young-history shock, so for each $s' \in \{h, l\}$,

$$c_m(K_t, s'; h) = c_m(K_t, s'; l). \tag{1}$$

Consider the middle-aged budget constraint at date $t+1$ conditional on $s_{t+1} = s'$. The only component that can differ across the two young histories $s_t \in \{h, l\}$ is inherited wealth from the young portfolio, which enters as

$$b_{y,t} + r(K_t, s')\, a_{y,t}.$$

Subtracting the middle-aged budget constraints across the two young histories and using (1) yields, for each $s' \in \{h, l\}$,

$$\Delta b_y + r(K_t, s')\, \Delta a_y = 0, \tag{2}$$

where $\Delta a_y \equiv a_y(K_{t-1}, h) - a_y(K_{t-1}, l)$ and $\Delta b_y \equiv b_y(K_{t-1}, h) - b_y(K_{t-1}, l)$. Equation (2) holds for $s' = h$ and $s' = l$. Subtracting the two equations gives

$$\big(r(K_t, h) - r(K_t, l)\big)\Delta a_y = 0.$$

If $r(K_t, h) \neq r(K_t, l)$, then $\Delta a_y = 0$, and substituting back into (2) implies $\Delta b_y = 0$. This proves the claim. $\square$

Lemmas 1 and 2 together characterize the restrictions imposed by short memory on individual behavior. Lemma 1 shows that strong stationarity mechanically erases cohort history from middle-aged decision rules, while Lemma 2 shows that, in the presence of state-contingent returns, this erasure forces young portfolio choices to be independent of the realized shock. These restrictions severely constrain how different cohorts can price assets at a given aggregate state. In particular, under short memory, a single one-period bond price must simultaneously satisfy the Euler equations of multiple cohorts whose wealth and consumption are shaped by different past decisions. The next proposition shows that these cross-cohort pricing requirements are generically incompatible.

Proposition 1 (Impossibility of short-memory equilibrium; genericity in C^2)**.** *Consider the three-period stochastic OLG economy with production described in Section 3. Fix the notion of short-memory competitive equilibrium in which equilibrium prices and allocations at date t depend only on the current aggregate state (K_{t-1}, s_t) (predetermined capital and current shock) and not on lagged endogenous variables. Then, for a generic (residual)*[1] *set of primitives (u, F) in the C^2 topology and for nondegenerate aggregate risk $\eta(h) \neq \eta(l)$, no short-memory competitive equilibrium exists. Equivalently, generically, imposing short memory yields incompatible cohort Euler conditions. Accordingly, no low-dimensional aggregate-state representation based only on (K_{t-1}, s_t) is generically possible, and the relevant equilibrium concept must allow for additional endogenous state variables.*

Proof sketch. The full derivation of the compatibility restrictions and the genericity argument are provided in the Appendices. Assume, toward a contradiction, that a short-memory competitive equilibrium exists.

[1] A property holds *generically* if it holds on a *residual* set: a countable intersection of open dense subsets of the relevant function space (here, the space of primitives endowed with the C^2 topology). The complement of a residual set is *meagre* (first category).

Step 1 (Implications of short memory). By Definition 1, equilibrium objects at date t depend only on the current aggregate state (K_{t-1}, s_t). Lemma 1 implies that middle-aged consumption is invariant to the young-history shock, while Lemma 2 shows that, under nondegenerate aggregate risk, young portfolio choices are independent of the realized shock.

Hence, under short memory, equilibrium allocations and portfolio choices satisfy strong cross-state restrictions.

Step 2 (Cross-cohort compatibility). At each current state (K_{t-1}, s_t), a single one-period bond price must satisfy the Euler equations of all active cohorts. Eliminating prices from these conditions yields the cross-cohort Euler compatibility restrictions derived in Appendix A.

Therefore, a short-memory equilibrium can exist only if these compatibility conditions hold at all states.

Step 3 (Generic failure). Appendix B establishes that the set of primitives for which these compatibility conditions hold is meagre in the C^2 topology.

Therefore, for a generic set of primitives, the compatibility conditions fail, implying that no short-memory equilibrium exists.

□

Intuition: failure of aggregation with capital. Proposition 1 can be interpreted as a failure of aggregation in a production economy. Although capital accumulation introduces a natural endogenous aggregate state variable, aggregate capital alone does not summarize the payoff-relevant distributional information needed for asset pricing. Different histories that lead to the same (K_{t-1}, s_t) can imply different inherited portfolio positions across cohorts, and therefore different marginal utilities and pricing kernels. As a result, a single price system depending only on (K_{t-1}, s_t) cannot satisfy all cohorts' Euler equations simultaneously. In particular, in contrast to exchange economies where the absence of a sufficient aggregate state variable is immediate, the production economy considered here contains an endogenous aggregate (capital) that is itself determined in equilibrium. Section 7 makes this intuition precise.

Compatibility restrictions (explicit two-state form)

The key implication of strong stationarity is that a *single* one-period bond price function must satisfy the bond Euler equation for *each* currently active cohort. In particular, at any current aggregate state, both the young and the middle-aged price the same one-period bond. Under short memory, the bond price depends only on the current state, so these Euler equations must be mutually consistent. Eliminating the bond price therefore yields necessary cross-cohort Euler compatibility restrictions for each current shock realization.

Let the current aggregate state at date t be $(K_{t-1}, s_t) = (k, s)$, where k is predetermined capital and $s \in \{h, l\}$ is the current shock. Under strong stationarity, next period's predetermined capital is pinned down by current portfolio choices, so define the induced short-memory law of motion

$$K^+(k, s) \equiv a_y(k, s) + a_m(k, s),$$

so that along an equilibrium path $K_t = K^+(K_{t-1}, s_t)$. For the two-step-ahead capital level, define

$$K^{++}(k, s, s') \equiv K^+\big(K^+(k, s), s'\big),$$

so that $K_{t+1} = K^{++}\big(k, s, s'\big)$.

Then, for each current shock $s \in \{h, l\}$ and each k in the equilibrium range, strong stationarity implies the compatibility restriction

$$\frac{u'\big(c_y(k, s)\big)}{u'\big(c_m(k, s)\big)} = \frac{\pi_h\, u'\big(c_m(K^+(k, s), h)\big) + \pi_l\, u'\big(c_m(K^+(k, s), l)\big)}{\pi_h\, u'\big(c_r(K^{++}(k, s, h), h)\big) + \pi_l\, u'\big(c_r(K^{++}(k, s, l), l)\big)} \tag{1}$$

Define from (1) the following residual function

$$\mathrm{CR}(k, s; u, F) := \frac{u'\big(c_y(k, s)\big)}{u'\big(c_m(k, s)\big)} - \frac{\pi_h\, u'\big(c_m(K^+(k, s), h)\big) + \pi_l\, u'\big(c_m(K^+(k, s), l)\big)}{\pi_h\, u'\big(c_r(K^{++}(k, s, h), h)\big) + \pi_l\, u'\big(c_r(K^{++}(k, s, l), l)\big)} \tag{2}$$

Then short memory requires $\mathrm{CR}(k, h; u, F) = \mathrm{CR}(k, l; u, F) = 0$ for every k in the equilibrium range. Appendix A derives these necessary restrictions from the fully expanded short-memory Euler equations; Appendix B proves that they are generically violated in the C^2 topology.

Benchmark cases in which short memory can survive

The incompatibility result in Proposition 1 relies on two ingredients: (i) some cohort makes portfolio decisions after the resolution of aggregate uncertainty, and (ii) asset returns are state contingent. These two ingredients generate multiple independent pricing restrictions that a single short-memory price system must satisfy simultaneously. Benchmark cases in which short memory can survive arise when this mechanism is shut down.

Proposition 2 (Benchmark cases with short memory)**.** *A short-memory equilibrium can be sustained in benchmark environments that eliminate the cross-state pricing conflict. In particular, this is the case (i) in a two-period OLG production economy with aggregate uncertainty, where no cohort re-optimizes portfolios after observing the current shock, and (ii) in the deterministic version of the present three-period model, where returns satisfy* $r(K_{t-1,}, h) = r(K_{t-1}, l) = r(K_{t-1})$ *and aggregate capital is a sufficient state variable for prices and allocations.*

Proof sketch. In case (i), no cohort makes portfolio decisions after the realization of the current shock, so the history-erasure and post-shock portfolio-rebalancing mechanism in Lemmas 1–2 is absent. Prices and allocations can therefore be written as functions of current capital and the current shock. In case (ii), the two linear restrictions in Lemma 2 collapse to the single equation $\Delta b_y + r(K_t)\Delta a_y = 0$ because returns are state independent. There is no overdetermination, no cross-state Euler incompatibility, and aggregate capital suffices to summarize equilibrium dynamics. These cases isolate the two distinct sources of the incompatibility: (i) post-shock portfolio choice, and (ii) state-contingent returns. Only when both are present does the short-memory obstruction arise. □

5. Recursive equilibrium and sequential market incompleteness

Since a short-memory equilibrium fails to exist generically, equilibrium must be represented using a richer state space. We therefore turn to recursive competitive equilibria in which additional endogenous state variables summarize payoff-relevant distributional information. I consider a time-homogeneous recursive equilibrium with state $x_t \equiv (K_{t-1}, s_t, \omega_t)$, where ω_t is a finite-dimensional vector of lagged endogenous variables sufficient to price assets and describe feasible trades. The additional state variable ω_t can be interpreted as a sufficient statistic for payoff-relevant distributional information, though it need not admit a low-dimensional representation.

5.1. Sequential market incompleteness

The role of ω_t is to summarize the payoff-relevant distributional information that cannot be inferred from predetermined capital and the contemporaneous shock alone. Once such an endogenous statistic is required for pricing, the next-period state $x_{t+1} = (K_t, s_{t+1}, \omega_{t+1})$ need not be determined solely by (K_t, s_{t+1}): post-shock portfolio rebalancing and state-contingent returns can generate multiple payoff-distinct realizations of ω_{t+1} with positive conditional probability, even holding fixed the same realization of s_{t+1}. Consequently, the conditional support of continuation states $\mathcal{S}(x_t)$ can naturally contain more than A elements, so that $J(x_t) = |\mathcal{S}(x_t)| \geq A + 1$ on a positive-probability set, which is exactly the environment in which sequential completeness fails by a dimension argument.

Sequential completeness is a state-by-state spanning condition: at a given recursive state, traded one-period securities allow agents to transfer wealth across all continuation states that occur with positive probability. Because the recursive state includes additional endogenous components, the set of continuation states can be large, and the following nondegeneracy condition ensures that this enlargement is payoff-relevant.

Assumption 1 (Nondegeneracy / rank condition). *Let $A := |\mathcal{A}|$ be the number of traded one-period securities. There exists a measurable set of recursive states $\mathcal{X}$ with $\Pr(x_t \in \mathcal{X}) > 0$ such that, for each $x \in \mathcal{X}$, the conditional support $\mathcal{S}(x)$ has cardinality $J(x) := |\mathcal{S}(x)| \geq A+1$, and the $J(x) \times A$ one-period return matrix*

$$R(x) := \big(R^a(x, x')\big)_{x' \in \mathcal{S}(x),\, a \in \mathcal{A}}$$

has rank A.

Given Assumption 1, we define sequential market completeness formally in terms of the span of one-period return vectors across the conditional support of continuation states.

Definition 2 (Sequential market completeness). *Fix a recursive state x. Let*

$$\mathcal{S}(x) \equiv \{x' : \Pr(x_{t+1} = x' \mid x_t = x) > 0\}$$

denote the finite conditional support of continuation states, and let $J(x) \equiv |\mathcal{S}(x)|$. For each traded one-period security $a \in \mathcal{A}$, define its one-period return vector *at x as*

$$R^a(x) \in \mathbb{R}^{J(x)}, \qquad R^a(x)_{x'} := R^a(x, x') \quad \textit{for each } x' \in \mathcal{S}(x),$$

i.e., the component $R^a(x)_{x'}$ is the gross payoff/return of asset a if the next state is x'. Markets are sequentially complete *at x if the return vectors of traded one-period securities span the entire contingent payoff space:*

$$\operatorname{span}\{R^a(x) : a \in \mathcal{A}\} = \mathbb{R}^{J(x)}.$$

Remark 1. *If $J(x) > |\mathcal{A}|$, then sequential completeness at x is impossible by dimension.*

The next lemma shows that once the recursive state generates more payoff-distinct continuation states than traded one-period securities, sequential completeness fails by a simple dimension argument.

Lemma 3 (Recursive state implies sequential incompleteness)**.** *Assume Proposition 1. Let $A := |\mathcal{A}|$ be the number of traded one-period securities. Suppose there exists a measurable set of recursive states $\mathcal{X}$ with $\Pr(x_t \in \mathcal{X}) > 0$ such that, for every $x \in \mathcal{X}$, the conditional support of continuation states satisfies $J(x) = |\mathcal{S}(x)| \geq A + 1$. Then markets are* not *sequentially complete at any $x \in \mathcal{X}$ in the sense of Definition 2.*

Proof. Fix $x \in \mathcal{X}$. By Definition 2, sequential completeness at x requires

$$\operatorname{span}\{R^a(x) : a \in \mathcal{A}\} = \mathbb{R}^{J(x)}.$$

However, the span of A vectors in $\mathbb{R}^{J(x)}$ has dimension at most A. Since $J(x) \geq A + 1$, it is impossible for A return vectors to span $\mathbb{R}^{J(x)}$. Hence markets are not sequentially complete at x. Because $\Pr(x_t \in \mathcal{X}) > 0$, incompleteness occurs on a positive-probability set of states. □

6. Welfare interpretation: interim efficiency and scope for improvement

This section clarifies the welfare meaning of the equilibrium-memory results without appealing to any specific policy instrument. The next proposition provides a welfare benchmark by linking sequential market completeness to short-run interim Pareto efficiency, thereby highlighting sequential incompleteness as a potential source of interim inefficiency.

Proposition 3 (Sequential completeness and short-run interim efficiency (Demange))**.** *If, at a recursive state x_t, markets are sequentially complete in the sense of Definition 2, then the competitive equilibrium allocation is short-run interim Pareto efficient at x_t.*

Proposition 3 is stated only as a benchmark for interpretation; see Demange (2002) for the proof. Short-run interim efficiency is the appropriate welfare notion in economies with finitely lived agents and sequential markets, as it evaluates Pareto improvements conditional on the information available at the current recursive state. Proposition 3 implies that whenever a recursive equilibrium is sequentially incomplete at a state, the equilibrium allocation need not be short-run interim Pareto efficient at that state. Combined with Lemma 3, this establishes that the equilibrium-memory requirement identified in this paper is generically associated with welfare losses, in the sense of foregone interim Pareto improvements.

7. Production and capital accumulation: failure of aggregation

Proposition 1 shows that a short-memory equilibrium representation is generically impossible. This section provides additional intuition for that result in the production economy by highlighting a failure of aggregation: even conditioning on predetermined aggregate capital and the contemporaneous shock, the information needed to price one-period assets need not be summarized by these aggregates. Post-shock portfolio rebalancing generates payoff-relevant cohort wealth differences that are invisible in the aggregate state but enter pricing kernels. Lemma 4 formalizes this mechanism by exhibiting histories that coincide in aggregate variables yet imply different marginal-utility-based pricing kernels, and hence different one-period prices.

Lemma 4 (Same aggregate capital, different pricing kernels)**.** *In the production economy, there exist histories that lead to the same current aggregate capital stock K_{t-1} and shock s_t but to different lagged cohort asset positions (and hence different ω_t). Along such histories, cohorts' current marginal utilities (and therefore one-period asset pricing kernels) differ, so equilibrium one-period prices cannot be functions of (K_{t-1}, s_t) alone.*

Proof. Let h^1 and h^2 be two histories satisfying $(K_{t-1}(h^1), s_t(h^1)) = (K_{t-1}(h^2), s_t(h^2))$ but $\omega_t(h^1) \neq \omega_t(h^2)$. By construction, ω_t summarizes payoff-relevant lagged portfolio/wealth information. Because middle-aged agents rebalance after observing s_t and current wealth includes inherited asset positions evaluated at state-contingent returns, the middle-aged budget set at date t differs across the two histories even though aggregate capital and the current shock coincide. Under strict monotonicity and concavity, the induced optimal current consumption choices differ for at least one active cohort, and thus current marginal utilities differ. The one-period stochastic discount factor relevant for asset pricing depends on these marginal utilities; hence equilibrium one-period prices differ across h^1 and h^2. Equivalently, the result can be interpreted as a failure to define a single stochastic discount factor consistent with all cohorts' marginal utilities when the state space is restricted to (K_{t-1}, s_t). Therefore equilibrium prices cannot be functions of (K_{t-1}, s_t) alone. $\square$

Remark 2 (Economic and technical role of production). *The introduction of production is not merely a change in environment, but is central to the contribution of the paper. Economically, capital accumulation creates a natural endogenous aggregate state variable that might have been expected to restore a low-dimensional equilibrium representation. The main insight of the paper is that this intuition fails: aggregate capital does not generically summarize the payoff-relevant distributional information needed for asset pricing across cohorts. Technically, production tightens the problem because current portfolio choices determine the future capital stock through the equilibrium law of motion, and future returns depend on that endogenous capital stock. Thus, the paper shows that the short-memory obstruction survives even in an environment with an endogenous aggregate state, which is precisely what distinguishes the production setting from the simpler exchange benchmark.*

8. Conclusion

The analysis highlights a fundamental limitation of aggregate-state representations in stochastic OLG economies with production. Even when a natural endogenous aggregate such as capital arises from equilibrium behavior, it need not suffice to summarize the information required for pricing and allocation. As a result, macroeconomic dynamics cannot, in general, be represented using a low-dimensional state space based only on aggregate variables. This underscores the role of payoff-relevant distributional information in environments with long-lived agents and aggregate risk, and suggests that richer recursive state spaces may be necessary in quantitative macroeconomic models that incorporate these features.

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Appendix A. Strong stationarity implies compatibility restrictions

Appendix A.1. Timing, history indexing, and proof of Lemma 1

Fix a cohort born at date t. Let $s_t \in \{h, l\}$ denote the shock realized when the cohort is young, s_{t+1} the shock when it is middle-aged, and s_{t+2} the shock when it is retired. At date $t+1$, when the cohort is middle-aged, the current aggregate state is (K_t, s_{t+1}).

Under short memory (Definition 1), middle-aged decision rules are functions of the current state only. Using the history-tag notation, we may write the middle-aged consumption chosen at $t+1$ as

$$c_{m,t+1} = c_m(K_t, s_{t+1}; s_t),$$

where the semicolon argument records the young-history shock. Short memory implies that this history tag is not payoff-relevant once the current state is fixed.

Proof of Lemma 1. Consider the middle-aged agent at date $t+1$ with state $(K_t, s_{t+1}) = (K, s')$ with a capital level K in the equilibrium range and $s' \in \{h, l\}$. By the short-memory restriction in Definition 1, the middle-aged policy is a function of the current state only; hence there exists a function $\widehat{c}_m(\cdot, \cdot)$ such that

$$c_m(K, s'; \sigma) = \widehat{c}_m(K, s') \qquad \text{for all } \sigma \in \{h, l\}.$$

Therefore $c_m(K, s'; h) = \widehat{c}_m(K, s') = c_m(K, s'; l)$, which proves Lemma 1. □

Appendix A.2. Euler equations and derivation of the compatibility restriction

Write prices at date t as $q_t \equiv q(K_{t-1}, s_t)$ and returns as $r_t \equiv r(K_{t-1}, s_t)$. Consider the young agent at date t with current state (K_{t-1}, s_t). The Euler equation for the one-period bond is

$$q(K_{t-1}, s_t)\, u'\big(c_y(K_{t-1}, s_t)\big) = \beta\, \mathbb{E}\Big[u'\big(c_m(K_t, s_{t+1})\big) \,\Big|\, s_t\Big]. \tag{A.1}$$

Similarly, for the middle-aged agent at date $t+1$ with state (K_t, s_{t+1}), the bond Euler equation is

$$q(K_t, s_{t+1})\, u'\big(c_m(K_t, s_{t+1})\big) = \beta\, \mathbb{E}\Big[u'\big(c_r(K_{t+1}, s_{t+2})\big) \,\Big|\, s_{t+1}\Big]. \tag{A.2}$$

Under the short-memory restriction, the next-period's predetermined capital stock is a function of the current state only. Define

$$K^+(K_{t-1}, s_t) \equiv a_y(K_{t-1}, s_t) + a_m(K_{t-1}, s_t), \qquad K^{++}(K_{t-1}, s_t, s_{t+1}) \equiv K^+(K^+(K_{t-1}, s_t), s_{t+1}).$$

Fix a current state (K, s) and evaluate (A.1) at $(K_{t-1}, s_t) = (K, s)$. Using i.i.d. shocks yields

$$q(K, s)\, u'\big(c_y(K, s)\big) = \beta\Big(\pi_h\, u'\big(c_m(K^+(K, s), h)\big) + \pi_l\, u'\big(c_m(K^+(K, s), l)\big)\Big). \tag{A.3}$$

Next, apply (A.2) at date $t+1$ and substitute $(K_t, s_{t+1}) = (K^+(K, s), s')$ to get, for each $s' \in \{h, l\}$,

$$q\big(K^+(K, s), s'\big)\, u'\big(c_m(K^+(K, s), s')\big) = \beta\Big(\pi_h\, u'\big(c_r(K^{++}(K, s, s'), h)\big) + \pi_l\, u'\big(c_r(K^{++}(K, s, s'), l)\big)\Big). \tag{A.4}$$

Evaluate (A.3) at the current state (K, s) and divide both sides by $u'(c_m(K, s))$:

$$q(K, s)\, \frac{u'(c_y(K, s))}{u'(c_m(K, s))} = \beta\, \frac{\pi_h\, u'(c_m(K^+(K, s), h)) + \pi_l\, u'(c_m(K^+(K, s), l))}{u'(c_m(K, s))}. \tag{A.5}$$

Next, apply (A.4) with $s' = s$ and rearrange to express $q(K, s)$:

$$q(K, s) = \beta\, \frac{\pi_h\, u'(c_r(K^{++}(K, s, h), h)) + \pi_l\, u'(c_r(K^{++}(K, s, l), l))}{u'(c_m(K, s))}. \tag{A.6}$$

Substituting (A.6) into (A.5) cancels the common factor $\beta / u'(c_m(K, s))$ and yields (2). Imposing this identity for both current shocks $s = h$ and $s = l$ is a knife-edge restriction on primitives; Appendix B shows that the set of primitives for which both restrictions hold is meagre, with residual complement.

Appendix B. Generic failure of short-memory compatibility restrictions

This appendix gives a compact argument that the compatibility restrictions induced by short memory are generically violated. Throughout, *generic* means *residual* (dense G_δ) in the C^2 topology on primitives. We endow the primitive space with the product C^2 norm, i.e. the supremum norm over functions and derivatives up to order two on the relevant compact domain. The argument identifies the minimal conditions under which cross-cohort Euler compatibility can hold and shows that such conditions fail robustly under arbitrarily small perturbations of primitives.

Appendix B.1. A residual map

Fix a compact set of capital levels $\mathcal{K} \subset (0, \infty)$ containing the relevant equilibrium range. For primitives (u, F) in a C^2 neighborhood satisfying the maintained monotonicity/concavity assumptions, consider the local candidate short-memory equilibrium objects $y(K; u, F)$ obtained by solving the equilibrium conditions other than the compatibility restrictions on a square subsystem. The resulting candidate map is assumed to be C^1 in (K, u, F) on the neighborhood under consideration.

Define the two-dimensional compatibility residual at each $K \in \mathcal{K}$ by

$$H(u, F)(K) \equiv \big(\mathrm{CR}(K, h; u, F),\, \mathrm{CR}(K, l; u, F)\big) \in \mathbb{R}^2,$$

where $\mathrm{CR}(K, s; u, F)$ comes from (2), the residual associated with (1), evaluated at the candidate short-memory objects $y(K; u, F)$. A short-memory equilibrium requires

$$H(u, F)(K) = 0 \quad \text{for all } K \in \mathcal{K}.$$

Appendix B.2. Local perturbations and the knife-edge property (IFT argument)

Fix primitives (u, F) and a point $K_0 \in \mathcal{K}$.

Admissible perturbations (openness). Let $\mathcal{P}$ denote the class of admissible primitives satisfying the maintained C^2 regularity, strict monotonicity, and strict concavity assumptions on the relevant compact domain. This class is open in the C^2 topology: if $u \in \mathcal{P}$, then every $\tilde{u}$ sufficiently close to u in the C^2 norm also lies in $\mathcal{P}$. Hence, by choosing perturbations sufficiently small, we can ensure that the perturbed utility u_λ remains admissible.

Existence of C^2 bump perturbations. We also use the standard bump-function fact: for any compact interval $I \subset (0, \infty)$ and any finite set of target points in I, there exist C^2 functions with arbitrarily small supports contained in I. In particular, for two distinct consumption levels $c_1 \neq c_2$ in the relevant range, we can choose $\varphi_1, \varphi_2 \in C^2$ with disjoint supports contained in small neighborhoods of c_1 and c_2, respectively. By scaling, we may normalize φ_1 and φ_2 so that

$$\|\lambda_1\varphi_1 + \lambda_2\varphi_2\|_{C^2}$$

is arbitrarily small for sufficiently small λ_1 and λ_2.

Consider a two-parameter family of C^2 perturbations of preferences

$$u_\lambda(c) = u(c) + \lambda_1\varphi_1(c) + \lambda_2\varphi_2(c), \qquad \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2,$$

where φ_1, φ_2 have small, disjoint supports chosen near generically distinct marginal-utility arguments entering the two residual components $\mathrm{CR}(K_0, h; \cdot)$ and $\mathrm{CR}(K_0, l; \cdot)$. For sufficiently small λ_1 and λ_2, the perturbed utility u_λ remains C^2, strictly increasing, and strictly concave on the relevant consumption range.

Now fix F and define the finite-dimensional map

$$G(\lambda) \equiv H(u_\lambda, F)(K_0) \in \mathbb{R}^2.$$

The Jacobian $DG(0) = D_\lambda H(u_\lambda, F)(K_0)|_{\lambda=0}$ is a 2×2 matrix. By the disjoint-support construction, φ_1 and φ_2 can be arranged so that the two residual components respond independently to (λ_1, λ_2); equivalently,

$$\det DG(0) \neq 0 \qquad (\text{so } \operatorname{rank} DG(0) = 2).$$

Implication (knife-edge). If $G(0) = 0$ and $DG(0)$ is invertible, the Implicit Function Theorem implies that the solution set

$$\mathcal{Z}_{K_0} \equiv \{\lambda : G(\lambda) = 0\}$$

is locally a 0-dimensional C^1 submanifold of $\mathbb{R}^2$ (equivalently, a codimension-2 set). In particular, $\mathcal{Z}_{K_0}$ has empty interior in a neighborhood of 0. Hence, for any neighborhood of (u, F) in the C^2 topology, there exist arbitrarily small perturbations (λ_1, λ_2) for which $H(u_\lambda, F)(K_0) \neq 0$. This shows that satisfying the two compatibility restrictions at K_0 is a knife-edge requirement.

Appendix B.3. From a pointwise knife-edge to a residual conclusion

Let $\{K_n\}_{n\geq 1}$ be a countable dense subset of $\mathcal{K}$. For each n, let

$$\mathcal{Z}_n \equiv \{(u, F) : \ H(u, F)(K_n) = 0\}.$$

By the continuity of the perturbation family in the C^2 topology and the local emptiness-of-interior result in the previous subsection, each $\mathcal{Z}_n$ is contained in a closed nowhere dense subset of the primitive space. Therefore the set

$$\mathcal{Z} \equiv \bigcap_{n\geq 1} \mathcal{Z}_n$$

is meagre (a countable intersection of nowhere dense sets). Its complement is residual. Since any (u, F) that satisfies $H(u, F)(K) = 0$ for all $K \in \mathcal{K}$ must in particular satisfy $H(u, F)(K_n) = 0$ for all n, the set of primitives admitting a short-memory equilibrium is contained in $\mathcal{Z}$ and is therefore meagre. Hence, for generic primitives, the compatibility restrictions fail and no short-memory equilibrium exists.

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