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A Theory of Payments-Chain Crises

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Abstract

This paper introduces an endogenous network of payments chains into a business cycle model. Agents order production in bilateral relations. Some payments are executed immediately. Other payments, chained payments, are delayed until other payments are executed. Because production starts only after orders are paid, chained payments induce production delays. In equilibrium, agents choose the amount of chained payments given interest rates and access to internal funds or credit lines. This choice determines the payments-chain network and aggregate total-factor productivity (TFP). The paper characterizes equilibrium dynamics and their innate inefficiencies. Agents internalize the direct costs of their payment delays, but do not internalize the costs induced onto others. This externality produces novel policy insights and rationalizes permanent reductions in TFP under excessive debt.

Key Words: Payments, Networks, Business Cycles

JEL Classification: E32, E42, G01

Says A: *I could use some of B's goods; but I have no cash to pay for them until someone with cash walks in here!*

Says B: *I could buy some of C's goods, but I've no cash to do it 'till someone with cash walks in here.*

From the book Stamp Scrip, Irving Fisher, 1933

1. Introduction

During financial crises, there are visible statistical declines in credit variables. Though harder to measure, there is also a general perception that the seamless flow of transactions of normal times slows down during crises. In ways that are yet to be better understood, there is also a concern that productive resources remain idle when agents have to wait longer to be paid and take longer to pay.

This paper proposes a model of payments chains and studies their implications in the context of financial crises. Providing a theory of payments-chain crises is important. Since the onset of modern business cycle analysis, economists have argued that TFP fluctuations reflect credit-market conditions (e.g., Summers, 1986). This view is even more salient during financial crises in developing economies. Economic contractions during these crises are predominantly driven by large declines in total factor productivity (TFP). These declines are explained by financial crisis models that stress input misallocation. However, producing large declines in total factor productivity remains challenging, given the smoother labor fluctuations observed during these episodes.[1] Payments-chain disruptions are an alternative mechanism to explain this co-movement. The slowdown of payments plays a central role in this alternative mechanism.

To model payments-chain disruptions and their effects on production, I introduce a payments-chain production network, in Section 2. A payments-chain network provides an explicit connection between the timing of economic transactions and the timing of production. In this network, production is organized through random bilateral relations where customers place production orders. Some orders, spot orders, are paid

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[1]The puzzle of large drops in TFP and capital utilization with small labor flows was first noted by Meza and Quintin (2005). The puzzle has been found in several contexts: Meza and Quintin (2007), Mendoza (2010), Oberfield (2013), or Karabarbounis et al. (2021), find large declines in TFP during the Chilean banking crisis of the early eighties, the Mexican, the East Asian sudden-stop crises of the mid-nineties, in the recent Greek crisis. These studies, in turn, cite many other examples.

upfront and their production begins immediately. Other orders, chained orders, are paid sequentially after other orders are paid. Since funds are transferred with delay, chained orders induce production delays.

A payments-chain network is a collection of payment chains that encompass the universe of economic transactions. Each payment chain is a sequence of linked transactions. The first payment of each chain is spot and, thus, executed with funds external to the network. All subsequent payments are chained and, therefore, executed with the funds from the previous transaction in its payment chain. An interpretation is that spot orders correspond to internal savings or working-capital lines, whereas chained orders represent production contracts.

An important feature is that the later the position in a payment chain, the longer the delay in producing an order. The ratio of chained orders to total orders characterizes the length-distribution of payment chains in the network. Measured average TFP is,

$$\mathcal{Y}\left(\mu;\delta\right) = \left(1-\mu\right) + \mu\frac{\delta - \delta\mu}{\mu - \delta\mu}\ln\left(\frac{1-\delta\mu}{1-\mu}\right) < 1,$$

where μ is the fraction of chained orders relative to total orders and δ is a parameter that captures production delays.

The payments-chain network of Section 2 is a stand-alone production block. This block is portable to other applications. In the second part of the paper, Section 3, I apply the theory to study business-cycle implications. To that end, I embed the payments-chain network into a tractable deterministic business-cycle model. In that setting, a payments-chain network is formed every period. There is a natural borrower and a natural saver which, for simplicity, I model as households. By construction, the saver always has funds to place spot orders. By contrast, the borrower carries outstanding debt. The borrower can obtain funds to place spot orders by borrowing from exogenous credit lines. However, her credit lines are limited by her outstanding debt. When outstanding debt impairs her access to short-term funds, she can still place chained orders. However, placing chained orders is privately costly, as goods bought through chained orders are more expensive. Critically, whereas borrowers internalize that chained orders are more expensive, they do not internalize that by placing chained orders, they delay other transactions.

The formula for TFP $\mathcal{Y}$ showcases how credit conditions translate into TFP declines. When short-term funds are limited, borrowers place chained orders to maintain a consumption level. This tightening of short-term funds increases the fraction of chained orders, μ. Measured TFP falls as labor inputs remain idle.

In the environment, the evolution of debt impacts TFP by impairing credit lines. In turn, TFP influences the desire to accumulate debt. There are three critical regions of the state space. For debt levels below an efficient-debt threshold, the economy is in a steady state without delays. For moderate debt levels, the economy features a *temporary* payments-chain crisis. Production delays last until the economy transitions toward an efficient steady state. For such moderate debt levels, crises are only temporary because borrowers have incentives to repay debts—to access short-term funding in the future. When debt is above a threshold, the economy features hysteresis, permanent payments-chain crises. Hysteresis occurs when borrowers decide to maintain high debt levels because the benefits of deleveraging happen to late into the future and require an excessive sacrifice of current consumption. The economy remains with permanent production delays in this debt overhang region of the state space.

The environment leads to novel policy implications presented in Section 4. I study a Ramsey planner that has instruments to influence the path of debt, but cannot tax spot and chained expenditures differentially. This Ramsey planner respects the payments technology but, as opposed to households, internalizes the effects of chained orders on TFP. The exercise illustrates that transitions are constrained inefficient for two reasons: savers spend too little in spot orders, and borrowers spend excessively in chained orders. Because the inefficiency is two-sided, debt may be excessively high or excessively low relative to the social optimum during a payments-chain crisis.

I also revisit fiscal multipliers. I allow the government to make spot expenditures or expenditures chained to tax receipts. Even though all forms of government spending are a waste, government spending can produce positive multipliers during a payments-chain crisis. A novel insight is that multipliers are positive only if the government makes spot expenditures. If government expenditures are chained to tax income, they are detrimental. The reason for positive multipliers is not the stimulus of aggregate demand, a conventional view regarding the benefits of fiscal policy. Rather, government

expenditures stimulate output by speeding up payments, a monetarist reinterpretation.

Literature Review.   The literature on financial crises is vast. This paper connects with theories that underscore the sharp declines in aggregate TFP. The link between financial crisis and TFP is not at all obvious because financial crises can manifest through distorting labor-market outcomes, not productivity. One branch of the literature explains declines in aggregate TFP through increased misallocation—see Pratap and Urrutia (2012) or Oberfield (2013). However, a common finding is that heterogeneity can only partially explain TFP declines once models are disciplined with data on input use and heterogeneity. Here, the channel is utilization. Other studies also explain declines in TFP as aggravated financial conditions increase the cost of utilizing capital—see Meza and Quintin (2007). The mechanism here is different: financial conditions impact TFP through the slowdown of payments.

Beyond the focus on financial crises, the paper falls at the crossroads of several areas. Namely, the monetary-payments literature, the economic-networks literature, and the literature on aggregate-demand externalities. The issue of how payments instruments affect production is a classic theme: Lucas and Stokey (1987) analyzes a stochastic cash-in-advance economy; Kiyotaki and Wright (1989) studies trade with indivisible tokens; Lagos and Wright (2005) a model with divisible money and explicit trading arrangements.[2] Recent work focuses on how the distribution, and not the instruments per se, affect production—see Lippi et al. (2015), Rocheteau et al. (2016), and Brunnermeier and Sannikov (2017). In common with this literature, the distribution of funding affects allocations. The main distinction is that I focus on delays in sequential payments.

Sequential payments appear in many other studies. The payments-chain network is inspired by the model in Kiyotaki and Moore (1997). Kiyotaki and Moore (1997) focus on finding micro-foundations of production disruptions in a single chain. The contribution here is to present a network of transactions whose structure depends on payment decisions. Furthermore, I embed the payments-chain network into a stan-

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[2]See Shi (1997); Lagos et al. (2011); Lagos and Rocheteau (2009); Li et al. (2012); Nosal and Rocheteau (2011); Rocheteau (2011) for many other directions in that area.

dard business-cycle model. Other models of sequential payments include Townsend (1980), which studies sequence of payments with spatial separation, Freeman (1996) and Green (1999), which study sequential transactions in an overlapping generation environments, La'O (2015), which studies a circular flow of transactions, or Guerrieri and Lorenzoni (2009), which studies sequential transactions in a Lagos and Wright-type environment. Recent work by Hardy et al. (2022) and Bocola (2022) contrast payments funded externally against trade credit.[3] Relative to these papers, there are two distinctions: here, transactions are formed in a network and the network is endogenous to expenditure-savings decisions. The importance of this body of theoretical work is substantiated by a body of empirical evidence found in a number of recent papers: Boissay and Gropp (2007), Jacobsen (2015), Barrot (2016), and Costello (2020) among many others.

With respect to the economic networks literature, the paper connects with models with endogenous network formation. The contribution relative to that literature is modest, as network formation is not strategic. By contrast, in Oberfield (2018), a network is formed through strategic partnerships. In Kopytov et al. (2022) and Elliott et al. (2022) firms form strategic links, being aware of possible supply-chain break-downs. Here, the network is randomly formed, but the distribution of chains is endogenous to financial decisions. Like in Elliott et al. (2014), Alvarez and Barlevy (2021) and Taschereau-Dumouchel (2022), there are externalities here too. In those models, externalities occur when individual defaults provoke subsequent defaults. Here, externalities occur through payment delays. Bigio and La'O (2013) considers the propagation of financial shocks that induces in misallocation in a production network.. Here, there is no misallocation in the production network, but there are production delays.

Finally, the paper connects with models of aggregate demand externalities. An early model of these externalities is Diamond (1982) where, via search, consumption decisions affect output. In most of the literature, demand externalities result from nominal rigidities. There has been a recent interest in coupling nominal rigidities with financial constraints—for example, Eggertsson and Krugman (2012) and Guerrieri and Lorenzoni (2017). Recent papers have further introduced sequential transactions into envi-

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[3]See also Biais and Gollier (1997).

ronments with nominal rigidities—for instance, Woodford (2022) and Guerrieri et al. (2022). In those models, demand externalities occur when agents cut back on any form of expenditure. The nature of demand externalities here is different. In particular, the type of expenditures by the private or public sector matters: spot orders may stimulate output, but chained orders depress it. The demand externalities provoked by the slow-down of payments is part of a classic narrative. Almost a century apart, the opening quotation taken from Fisher (1933) and the renowned "babysitting co-op" analogy of Krugman (1998) belong to that same tradition. The rest of the paper is an attempt to provide an analytic formulation.

2. Payments-Chains and Productivity

This section presents the payments-chain network. I then embed it into a dynamic business-cycle model.

Bilateral Relations. Production is organized through bilateral agreements in which a customer orders a product from an agent that owns a production unit. The agreement is exclusive in that only the agent placing the order can derive utility from its production. In turn, production units are exclusively dedicated to producing for a specific client.[4] As examples of such relations, we can think of a home renovation project, a medical service, the manufacturing of an engineered product with a specific blueprint, or the commission of a piece of art.

There are two types of orders; *spot* and *chained* orders. Spot orders are paid immediately. Chained orders are paid after the client receives a payment from another transaction. There are N production units. In turn, there are N^s spot orders and N^x chained orders. There are equal numbers of production units and orders, $N = N^s + N^x$. In the following section, I work with a limit for $N \to \infty$ and recast this condition as a market-clearing condition. Each production unit is assigned an identifier, $i \in \mathcal{N} = \{1, 2, \ldots, N\}$. Likewise, each order is assigned a unique identifier, $i \in \mathcal{N}$. I partition $\mathcal{N}$ into two sets, $\mathcal{N}^s$ and $\mathcal{N}^x$, to denote the set of identifiers of spot and chained orders,

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[4]The agents' identity in these agreements is not explicit at this stage. Identities are explicit and these matter for the economy's dynamics in the following section.

respectively.[5] Also, I work with the assumption that payments are identical across orders, a condition that I explain in the following section. I also provide an interpretation of spot and chained once I introduce financial decisions in that section.

I define two relations that together define the payments-chain network. First, $\mathcal{P} : \mathcal{N} \to \mathcal{N}$, is a one-to-one assignment from an order to a production unit. By assumption $i \neq \mathcal{P}(i)$. The interpretation is that $\mathcal{P}(i)$ is the unit that produces order i. The assignment $\mathcal{P}$ is entirely random: any unit can be assigned to any order with equal probability. Second, a chained income-expenditure relation associates a chained order to a production unit. This relation is the identity function defined on $\mathcal{N}^x$. That is, order i uses the revenues of production unit i if $i \in \mathcal{N}^x$, to make payments. The idea is that although order i is not externally funded, the customer that places order i owns production unit i. Hence, order i can be funded after order $j = \mathcal{P}^-(i)$ executes a payment. The assumption that the income-expenditure relation is the identity is innocuous.[6]

To anticipate how the two relations induce a payments-chain network, consider production unit j assigned to order i, $\mathcal{P}(i) = j$. The client placing order i must pay for $j's$ production. This creates a payment link from i to j. In turn, if order j is spot, $j \in \mathcal{N}^s$, the funds paid in order i are not used in further payments. However, if order j is chained, the funds are used again to pay unit $k = \mathcal{P}(j)$. In other words, when $j \in \mathcal{N}^x$, there is a flow of payments from i to j, and from j to k. If order k is also chained, the same funds are used to pay unit $\mathcal{P}(k)$, and so on. The payments chain continues until a final order in the chain is placed on some production unit i not associated with a chained order. Since every order is paired with a production unit, the economy features an entire network of transactions that forms a collection of payment chains.

The payments-chain network determines production. The production of orders occurs within a unit time interval. Each order starts at some time $\tau \in [0, 1]$. Once production starts, it cannot stop. Production is linear in time: if production of starts at τ, production is $1 - \tau$.

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[5]$\mathcal{N}^s$ and $\mathcal{N}^x$ are a partition of $\mathcal{N}$: $\mathcal{N}^s \cap \mathcal{N}^x = \emptyset$ and $\mathcal{N}^s \cup \mathcal{N}^x = \mathcal{N}$. The number of elements is $N^s = \#\mathcal{N}^s$, $N^x = \#\mathcal{N}^x$.

[6]Formally, we can define the chained income-expenditure relation $\mathcal{X}$ as the identity function on $\mathcal{N}^x$, that is $\mathcal{X} : \mathcal{N}^x \to \mathcal{N}^x$ such that $\mathcal{X}(i) = i$. The idea is that $i \in \mathcal{N}^x$ obtains funds from production unit, $\mathcal{X}(i) = i$. Indeed, the identity function $\mathcal{X}(i)$ can be replaced by any injective function $\mathcal{X} : \mathcal{N}^x \to \mathcal{N}$ so that production units and associated chained orders do not have the same identifiers. Changing identifiers does not affect the results.

If every order were to start immediately, production would be maximal. Maximal production may fail because of two frictions that make the payments-chain network consequential. The first friction is that production can only start once there is proof of funds. If an order is spot, by definition, proof of funds is immediate since the order is funded externally. By contrast, chained orders can prove funds only after the payment of the order that is its source of funds has been executed. Production requires proof of funds to deter fraudulent behavior: without proof of funds, clients could promise payments that they know will never occur. Since production is customized, this would lead to a disadvantageous ex-post renegotiation.

If all funds could be transferred instantaneously, all orders would start immediately. All that would be needed is have each spot order make an initial payment, and funds would reach each chained order instantaneously. If that were the case, the payments-chain network would be inconsequential. The second friction, limited commitment, provokes a delay in fund transfers. The idea is that after the customer proofs funds to start an order, the funds are only released after the fraction $1 - \delta$ of order's output is inspected. Without inspection, the producer could produce another good to his advantage. Since there is no way to verify a customization, the inspection is necessary to avoid moral hazard.

Placing these frictions together, we obtain a production structure where the greater the number of chained orders, the more payment delays and the longer the production delays. To produce predictions about aggregate TFP, I first formalize the definition of the payments-chain network.

Payments-Chain Network. A sequence of payments with a single original source of funds defines a **payments chain**. Naturally, every chain must start with a spot order, followed by a sequence of chained orders in which those initial funds are passed along from agent to agent. The number of chained orders defines the **chain length**. A **payments-chain network** is the collection of all payment chains, the universe of transactions during the production time interval. I employ a formal definition.

Definition 1. *A **payments-chain network** $\mathcal{K}$ is an acyclical directed network with nodes $\mathcal{N} = \{1, 2, ..., N\}$ and links $\mathcal{V} = \{(i, j) \, | \, \mathcal{P}(i) = j, j \in \mathcal{N}^x\}$, $\mathcal{K} = (\mathcal{N}, \mathcal{V})$. A **payments-***

chain *of length n is a finite sequence of nodes $\{i_k\}_{k=0}^{n}$ such that the sequence starts with some $i_0 \in \mathcal{N}^s$ and $\forall k \in \{1, \ldots, n\}$, $\mathcal{P}(i_{k-1}) = i_k \in \mathcal{N}^x$. By convention, if $i \in \mathcal{N}^s$ and $j = \mathcal{P}(i) \notin \mathcal{N}^x$, then i and j define a chain of length zero.*

Nodes corresponding to the payments-chain network represent, both, orders and production units. The directed links represent the direction of the flows of funds. Because production is bilateral, one or no links stem from each node. A link from i to j indicates that i orders from production unit j and that j is a chained order, $\mathcal{P}(i) = j \in \mathcal{N}^x$. The source of funds of order j is the funds paid in order i. In turn, if a node does not receive a link, it represents a spot order. Furthermore, if no links are directed toward order $j = \mathcal{P}(i)$, then order j is also spot—in which case i and j form a zero-length chain. Notice that in this construction, each order has a source of funds. Any (longest) path of links defines a payments chain. The collection of payment chains is the payments-chain network.

Examples. Let me provide an example. Set $\mathcal{N} = \{1, 2, \ldots, 8\}$ and let the subset of spot orders be $\mathcal{N}^s = \{1, 3, 7\}$. Also, define the production relation as follows: let $\{i_n\}_{n \in \mathcal{N}} = \{1, 5, 7, 4, 6, 2, 3, 8\}$ such that $i_{n+1} = \mathcal{P}(i_n)$ and $i_1 = \mathcal{P}(i_N)$. Thus, the links in this payments-chain network are $\mathcal{V} = \{(1, 5), (7, 4), (4, 6), (6, 2), (3, 8)\}$.

Several graphs are associated with this example. The left panel of Figure 1 depicts the chained income-expenditure relation. In that panel, I split each node into counterparts: the production units $\{u_n\}$ and production orders $\{o_n\}$ for nodes, $n \in \{1, 2, ..., 8\}$. The links represent the flow of funds from production units to their corresponding chained orders, defined by the chained income-expenditure relation.

The middle panel adds the links to the flow of payments for production, corresponding to $\mathcal{P}$. That is, the links in that panel add the payments production units. Adding the links of the chained income-expenditure and production relations allows us to trace funds. Notice that the links from orders to production units with the same color share a common source of funds.

The right panel depicts the payments-chain network. In this example, there are three chains. The first chain is of length one—from 1 to 5. The second chain starts at 7, and links nodes 7, 4, 6 and 2. Since 4, 6 and 2 are chained orders, the length of the

second chain is three. The third chain links node 3 with 8 and is also of length one.

(a) Income relation   (b) Network in detail   (c) Payments-Chain Network
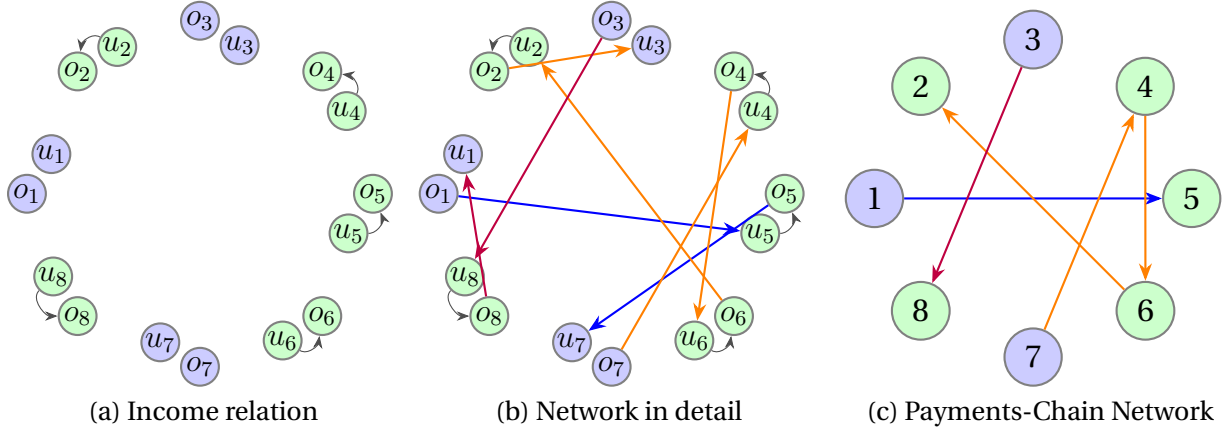
Figure 1: Components of the Payments-Chain Network

Average TFP. I now derive the distribution of payments chain lengths as the number of transactions increases, $N \to \infty$. As the number of transactions increases, I keep the ratio $\mu \equiv N^x/N$ constant. This fraction is a parameter of the probability distribution $G(n; \mu)$ chain length in the network.

Proposition 1. Let $n \in \{0, 1, 2, \ldots\}$ be the length of a payment chain in the payments-chain network. Then, n is a random variable with probability mass function (p.m.f.) $G(n; \mu)$ where $G(n; \mu)$ is the *geometric p.m.f.* with parameter μ, that is, $G(n; \mu) = (1 - \mu)\,\mu^n$.

Proof. Recall that the production relation is random. As $N \to \infty$, a node has a link directed toward it if it corresponds to a chained order. Thus, a node is linked with probability μ. A node does not receive a link if it is spot, with probability $1 - \mu$. Furthermore, recall that each chained order is funded and thus belongs to specific payment chain which, in turn, starts with a specific spot order. Thus, there is a one-to-one relation between each payment chain and a spot order. Hence, there are N^s chains in total, which we can also index by $i \in \mathcal{N}^s$. If chain i forms a chain of length zero, it must be that $\mathcal{P}(i) \in \mathcal{N}^s$. This, happens with probability $1 - \mu$—because each production link

indeed occurs with equal probability. Likewise, chain i is of length 1 if $\mathcal{P}(i) \in \mathcal{N}^x$ but $\mathcal{P}(\mathcal{P}(i)) \notin \mathcal{N}^x$. This happens with probability $\mu \times (1 - \mu)$. In a chain of length two, there are two consecutive chained orders followed by a specific spot order. This occurs with probability $\mu^2(1 - \mu)$. Proceeding by induction, we arrive at the geometric distribution.

$\square$

I use this distribution to obtain aggregate production: I first derive the production in a chain of arbitrary length. In a chain of length zero, there is always a spot order whose production begins immediately. In a chain of length one, the production of the first spot order begins immediately, but there is a delay in the second order. For the second order, the funds are received after $1 - \delta$ of the production of the first order is finished. This happens at time $\tau = 1 - \delta$. Hence, this leaves only δ time to produce the second order in the chain. For the second order, production is δ of which the $1 - \delta$ fraction must be inspected. If there is a third order in the chain, the transfer of funds occurs $(1 - \delta)\delta$ time after the first transfer at $1 - \delta$. Adding these consecutive delays, the production of the third order can only start by $(1 - \delta) + \delta(1 - \delta) = 1 - \delta^2$. This leaves δ^2 time to produce that third order.

We can deduce a pattern by forward induction.[7] In a chain of length n, the corresponding production vector in of $n + 1$ consecutive orders in a chain is $\{1, \delta, \delta^2, \ldots, \delta^n\}$. Using the distribution of chain lengths to compute expectations, I obtain aggregate production.

Proposition 2. (Output per worker): *Given μ and δ, the average output corresponding to chained orders converges to:*

$$\mathcal{A}(\mu; \delta) = \frac{1 - \mu}{\mu} \frac{\delta}{1 - \delta} \ln\left(\frac{1 - \delta\mu}{1 - \mu}\right) < 1. \tag{1}$$

as $N \to \infty$. $\mathcal{A}$ is strictly concave, decreasing, and satisfies

$$\lim_{\mu \to 0} \mathcal{A}(\mu; \delta) = \delta \text{ and } \lim_{\mu \to 1} \mathcal{A}(\mu; \delta) = 0 \text{ and } \lim_{\delta \to 0} \mathcal{A}(\mu; \delta) = 0 \text{ and } \lim_{\delta \to 1} \mathcal{A}(\mu; \delta) = 1.$$

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[7]If the k-th node initiates production at time $1 - \delta^{k-1}$, the delay from its inspection is $(1 - \delta)\delta^{k-1}$, which added to previous delays leads to a transfer of funds only by time $1 - \delta^{k-1} + (1 - \delta)\delta^{k-1} = 1 - \delta^k$. This, leaves δ^k time for production to the subsequent unit. Since we computed the delay for $k = 1, 2, \ldots$. the productions in a chain of length n are $\{1, \delta, \delta^2, \ldots, \delta^n\}$.

Average output is $\mathcal{Y}(\mu) = (1 - \mu) + \mu\mathcal{A}(\mu) \leq 1.$

This theorem is key as it presents a formula for output. To obtain output, I first compute the average production among chained orders, $\mathcal{A}(\mu; \delta)$.[8] $\mathcal{A}$ is strictly concave and decreasing in μ and has well-behaved limits.[9] Total output $\mathcal{Y}(\mu; \delta)$ is constructed by noting there are $(1 - \mu)$ spot orders for which production is 1 and μ for chained orders whose average production is $\mathcal{A}$. Since labor is in fixed supply, $\mathcal{Y}$ is both output and TFP. However, since efficiency losses are entirely driven by utilization among chained orders, I refer to $\mathcal{A}$ as TFP directly. To illustrate the formula for $\mathcal{A}(\mu; \delta)$ and its construction, the left panel of Figure 2 plots the distributions of chain lengths—corresponding to two values of μ. For the higher μ, the distribution shifts mass to chains of greater length. The right panel graphs $\mathcal{A}$ as a function of μ, for two values of δ. Productivity falls as the fraction μ increases. In turn, the lower δ, the greater the delays and the lower TFP.

The main insight is that TFP losses result from payment delays. It is worth distinguishing the sources of these losses from other environments. Unlike search models, there are no congestion externalities—the production assignment s one-to-one. Unlike sticky-price models, the fact that payments are the same across orders does not have implications for production (input uses do not depend on prices). The source of TFP losses are the production delays caused by payment delays. The economy is at full capacity when if all transactions are spot. Importantly, TFP losses are magnified by the random assignment of orders. Holding fixed the number of spot orders, if pay-

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[8]To obtain output, I first compute the average production of chained orders in an n-length chain, $\left(n^{-1}\sum_{i=1}^{n}\delta^{i}\right)$. With the average production of each chained orders, I can compute the expected value of production in chains with at least one chained order. Integrating across all possible lengths, we obtain $\mathcal{A}(\mu; \delta)$, the average production for chained orders. For that, I use the discrete probability $(1 - \mu)\mu^{n}/\mu$. This is the distribution of chain length conditioned on $n > 0$, obtained from $G(n; \mu)$. Integrating across all possible lengths, we obtain $\mathcal{A}(\mu; \delta)$, the average production for chained orders.

[9]With respect to the limits of $\mathcal{A}$, as $\mu \to 1$ the chains become larger but their average production decreases to zero since the additional production of chained order decreases exponentially. On the other hand, when $\mu \to 0$ the chain length tends to 1 and thus per-worker productivity tends to δ since average delay times converge to one period. In turn, when $\delta \to 0$ transfers take the full production period leaving not time for production. Conversely if $\delta \to 1$, chained orders feature no delays. Interestingly, $\mathcal{A}(\mu; \delta)$ resembles an entropy function. I am unaware of a connections between the geometric distribution, a discounted sequence, and the entropy function.
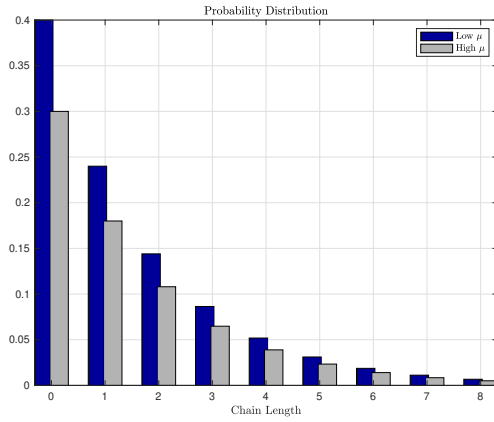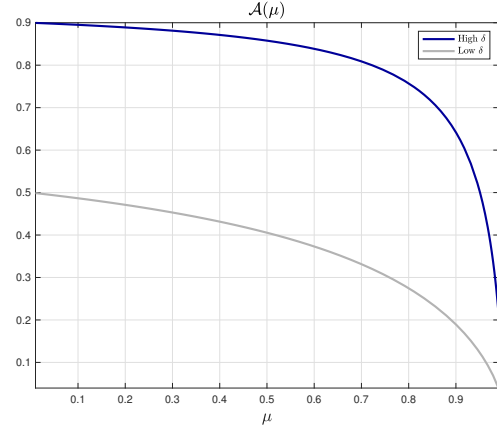
(a) Chain length distribution   (b) TFP

Figure 2: Chain-Length Distributions and TFP

Note: Left panel graphs $G(n)$ for $\mu = \{0.6, 0.7\}$. Right panel $\mathcal{A}(\mu)$ for $\delta = \{0.5, 0.9\}$.

ment chains could be reorganized to be all of the same length, the economy would still feature TFP losses, but less so than under random assignment (Jensen's inequality).

Discussion: Reduction of Economic Complexity. The payments-chain network here is simple. In practice, economies involve more complexity than exhibited here: Transactions differ by size and are coupled in more intricate production networks. Moreover, production ends by the end of the period and orders are not withdrawn.[10] Studying these richer dimensions would make the problem more realistic, but complicate the analysis.[11] Despite the crude assumptions, the payments-chain network illustrates the effects of delays. A virtue is that we can obtain a mapping from a financial quantity, μ, to measurable TFP. Because μ depends on agent decisions in the following section, we can study policy exercises immune to the Lucas critique.

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[10]To allow withdrawals, we would have to find an endogenous maximal chain-length provided that we obtain a relative price for funds and production, as I do in the following section. In that case, the production in a chain would drop to zero for orders above a given position. An analog, but more complicated, formula to TFP, (1), can be derived for the case.

[11]Statistical physics handles combinatorial problems that lead to this complexity. Potentially, such methods can be used in a similar environment to the one here. In particular, statistical physics offers tools to calculate statistical properties of the system.

3. Payments-Chains in a Business-Cycle Model

I now incorporate the payments-chain network into a business cycle setting. I use the results of the previous section as inputs.

3.1 Environment

Timing. The horizon is infinite. Expenditure and savings decisions are programmed at integer dates, $t = \{0, 1, 2, ...\}$. Production happens within the time interval between integer dates, according to the payments-chain network of the previous section. It is no longer necessary to refer to the time interval of production; it is understood to happen within dates. There is perfect foresight. Labor units are the numeraire.

Demographics. The economy is populated by two households. One is the worker household that inelastically supplies labor and starts with negative financial wealth. The other is the saver household that has positive financial wealth but does not produce. Each household consumes goods by placing production orders in the payments-chain network. I use superscripts s and w to distinguish saver from worker variables. The advantage of this setup is that it allows for closed forms, akin to endowment economies with log-utility, while allowing for labor income.

Income, Expenditures, Transactions, and Prices. Production and consumption depend on the expenditure mix between spot and chained orders by both households. The worker household is endowed with N labor units. As in the previous section, a labor unit is assigned to a single order. I normalize payment per labor unit to $1/N$ so that total labor income is one.

I work with the limit as $N \to \infty$. As I increase the number of orders, I scale the production of each labor unit by $1/N$, to keep maximal feasible production equal to one. Implicitly, households place a large number of orders. Aggregating across all those orders, the risk in the position of a chain order is diversified. This assumption is akin to the classic "big-family" assumption of Lucas and Stokey (1987). In the previous section, I assumed that all production units are paid the same, regardless of their production. A

motivation for this assumption is pairwise stability.[12]

At any t, both households choose amounts of spot and chained orders. To place spot orders, households must possess funds. Savers do not produce, so they are restricted to only making spot payments. For the rest of the paper, it is understood that the saver's period consumption C^s is purchased via spot orders. By contrast, the worker household has to choose between spot and chained orders. The total of goods purchased by workers is C^w

$$C^w = S^w + X^w, \tag{2}$$

where S^w and X^w are her goods purchased via spot and chained orders (henceforth, spot and chained goods respectively). As in the previous section, each chained order must be backed by a labor income unit.

Aggregating across households, there is a total of S spot goods and X chained goods produced. In turn, adding the expenditures of both households, expenditures in spot and chained orders are E^s and E^x, respectively. In equilibrium, the total income (of households) must equal total expenditures, so $E^x + E^s = 1$. The ratio of chained expenditures to total expenditures is $\mu = E^x$ and, thus, $E^s = 1 - \mu$.

In the previous section, we worked with integer amounts of production units and orders. I further imposed the countability condition, $N = N^s + N^x$. Here, expenditures are not restricted to be natural numbers when households decide on them. Moreover, I impose a labor market-clearing condition. Both properties are internally consistent. To see this, we can count the number of spot and chained orders with a floor and ceiling function, $N^s = \lfloor E_t^s \times N \rfloor$, $N^x = \lceil E_t^x \times N \rceil$, given expenditure choices. The countability condition is satisfied as long as $E^x + E^s = 1$; any inconsistencies—due to rounding errors—between the expenditure choices and the number of goods bought vanish as $N \to \infty$.

Recall that total output depends on μ. Since for each spot order, there is one unit of output, $E^s = S = 1 - \mu$. Thus, using Theorem 2 we have that $\mu \mathcal{A}(\mu) = X$. If we

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[12]See Bloch and Jackson (2006) for a definition. Applied here, pairwise stability requires agents to accept the links in a network at the moment of placing orders—without knowledge of the location in the network. To accept an order, labor units must be paid the same. Otherwise, they would sever links and a new network would dissolve. In tun, to place an order, households must be willing to do so, anticipating the average amount of goods obtained by placing spot and chained orders, respectively. This is the case in this model.

substitute $\mu = E^x$ into this relation, we find that $E^x = \mathcal{A}^{-1}(\mu) X$. Because the number of orders tends to infinity, we can treat $q(\mu) \equiv \mathcal{A}^{-1}(\mu)$ as a price of chained goods per unit of chained expenditure. I use this auxiliary price, to define a worker expenditure bundle:

$$S^w + q(\mu) X^w = E^w. \tag{3}$$

where E^w are the worker expenditures.

Savers. The saver's period utility is $\log(\cdot)$. He maximizes discounted lifetime utility over the sequence $\{C_t^s\}$. Savers begin each t with real deposits, D_t, their only source of wealth. Deposits earn an equilibrium return R_t. Given D_t, savers choose future savings, D_{t+1}, and current expenditures.

Problem 1. (Saver Problem): *Given D_0 and $\{R_{t+1}\}_{t\geq 0}$,*

$$\max_{\{C_t^s\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \log(C_t^s),$$

subject to the budget constraint, $R_{t+1}^{-1} D_{t+1} + C_t^s = D_t, \forall t \geq 0$.

Workers. The worker's preferences are the same. Different from savers though, workers begin each t with debt, B_t, and choose between current expenditures and future debt B_{t+1}. The choice B_{t+1} is limited by a natural debt limit, $\bar{B} = 1/(1-\beta)$. Because they are in debt, the worker must borrow intra-period to make spot expenditures, . Namely, she comes with B_t, but increases her debt to $B_t + S_t^w$ at the start of the period. This intra-period debt carries no interest. Its repayment is always feasible since total labor income always exceeds worker expenditures—because saver and worker expenditures add up to labor income in equilibrium.By the end of the period, intra-period debt is either paid or added to the balance of B_{t+1}, depending on the worker's overall expenditures. Critically, intra-period debt is limited by a time-varying spot-borrowing line (SBL), $\tilde{B}_t$:

$$S_t^w \leq \bar{S}_t \equiv \max\left\{\tilde{B}_t - B_t, 0\right\}. \tag{4}$$

An interpretation of $\tilde{B}_t$ is that it is a credit line that caps the amount of intra-period borrowing by $\bar{S}_t$.

The worker can consume without placing spot orders by placing chained orders. However, chained orders are costlier because $q \geq 1$.

Problem 2. (Worker Problem): *Given* B_0 *and* $\left\{ R_{t+1}, \tilde{B}_t \right\}_{t \geq 0}$,

$$\max_{\{S_t^w, X_t^w\}_{t \geq 0}} \sum_{t \geq 0} \beta^t \log\left(C_t^w\right),$$

subject to the budget constraint, $B_t + E_t^w = R_{t+1}^{-1} B_{t+1} + 1$, $\forall t \geq 0$, *to the expenditure mix (3) and total consumption (2), to the intra- and inter-period constraints (4), and to the natural debt limit* $B_t \leq \bar{B}$.

Clearly, the worker will always prefer to make spot payments. However, if the worker has little intra-period borrowing capacity she has to make costly chained expenditures to reach a desired level of consumption.

Market Clearing. Clearing in the asset market requires:

$$D_t = B_t. \tag{5}$$

Recall that since savers do not work, they must always maintain positive savings, $D_t > 0$. Thus, without loss of generality, I work under the assumption that the worker is always in debt. Hence, from now on, I no longer refer to D_t, and use B_t to represent both saver deposit and worker debt. Given the consumption choices of both households, $\{X_t^w, S_t^w, C_t^s\}$, the goods-market clearing condition is:

$$C_t^s + S_t^w + X_t^w = \mathcal{Y}\left(\mu_t\right). \tag{6}$$

Adding both household's budget constraints yields an income-expenditure identity:

$$C_t^s + S_t^w + q_t X_t^w = 1. \tag{7}$$

Definition 2. *Given a sequence of* $\left\{\tilde{B}_t\right\}_{t \geq 0}$, *a sequence* $\{B_t, C_t^s, S_t^w, X_t^w, R_t, q_t\}_{t \geq 0}$ *is a* ***symmetric competitive equilibrium*** *if:*

1. *Given* $\{R_t, q_t\}_{t \geq 0}$, $\{B_t, S_t^w, X_t^w\}_{t \geq 0}$ *solves the worker's problem and* $\{B_t, C_t^s\}_{t \geq 0}$ *solves the saver's problem.*

2. *Markets clear; (5) and (6) are satisfied.*

In equilibrium, the ratio of chained expenditures satisfies the following payments identity:

$$\mu_t = q\left(\mu_t\right) \cdot X_t^w. \tag{8}$$

3.2 Characterization

Household problem solutions. The solution to the saver's problem is typical of log utility and goes without proof.

Proposition 3. *Given* B_0, *the solution to the saver's problem is:*

$$C_t^s = (1 - \beta) B_t \ \text{ and } B_{t+1} = R_{t+1} \beta B_t \ \forall t \geq 0. \tag{9}$$

The worker's problem is more involved. Given total expenditures E_t^w, since $q_t \geq 1$, cost minimization requires the worker to spend spot as much as her SBL allows. Thus, given E_t^w, spot and chained expenditures are respectively:

$$S_t^w = \min\left\{\bar{S}_t, E_t^w\right\} \tag{10}$$

and

$$X_t^w = \left(E_t^w - \min\left\{\bar{S}_t, E_t^w\right\}\right) / q_t. \tag{11}$$

Given this optimal split, I invoke the principle of optimality to cast the worker's problem into a Bellman equation:

Problem 3. (Worker (Recursive) Problem): *Given* B_0 *and* $\left\{\tilde{B}_t, R_{t+1}, q_t\right\}_{t \geq 0}$, *workers choose a sequence of debt holdings* $\{B_{t+1}\}_{t \geq 0}$ *which follow from the solution to:*

$$V_t^w(B) = \max_{B' \leq \bar{B}} \log\left(S^w + X^w\right) + \beta V_{t+1}^w(B') \tag{12}$$

where S^w and X^w are given by (10) and (11), respectively, and total expenditures by $E^w = B'R_{t+1}^{-1} + 1 - B$.

The time index in the value function reflects the time dependence on R_{t+1}, q_t, and $\tilde{B}_t$. The Bellman equation reformulates the worker's problem as an expenditure-savings problem. The solution yields total expenditures. Using total expenditures as inputs to the optimal expenditure rules, (10) and (11), we obtain optimal spot and chained consumption. The Bellman equation is key to understand the business-cycle implications of payments-chain networks. The following lemma identifies two threshold points that characterize the worker's expenditure mix in terms of their debt.

Lemma 1. (Expenditure Threshold Points): *Define the **efficiency threshold**, $B_{t+1}^\star \equiv R_{t+1}\left(\tilde{B}_t - 1\right)$. Then, $S_{t+1}^w = 0$ if and only if $B_{t+1} > \tilde{B}_{t+1}$. In addition, $X_t^w > 0$ if and only if $B_{t+1} > B_{t+1}^\star$.*

Proof. The proof is immediate. If $B_{t+1} > \tilde{B}_{t+1}$, the worker cannot spend spot. In turn, $B_{t+1} > R_{t+1}\left(\tilde{B}_t - 1\right)$ happens if and only if $E_t^w = B_{t+1}/R_{t+1} - B_t + 1 > \tilde{B}_t - B_t$. In that case, the worker spends sufficiently high that his chosen expenditure E_t^w requires him to spend on some chained orders.

$\square$

It is convenient to define some relevant objects derived from these threshold points.

Definition 3. (Marginal Prices and Marginal inflation):

I. *The **average price** per unit of worker expenditure is $Q_t \equiv E_t^w/C_t^w$.*

II. *Given B', the price of the good at t bought with a marginal increase in B' is:*

$$\tilde{q}_t^E\left(B'\right) \equiv 1 + \left(q_t - 1\right)\mathbb{I}_{\left[B' \geq B_{t+1}^\star\right]},$$

the price of the good purchased at $t+1$ after a marginal decrease in B' is

$$\tilde{q}_{t+1}^B\left(B'\right) \equiv 1 + \left(q_{t+1} - 1\right)\mathbb{I}_{\left[B' > \tilde{B}_{t+1}\right]}.$$

*These terms define **marginal prices**.*

III. ***Marginal inflation*** *is* $\Pi_{t+1}(B') \equiv \tilde{q}^B_{t+1}(B')/\tilde{q}^E_t(B')$, *a continuous function of* B'
   except at discontinuity points $\left\{B^\star_{t+1}, \tilde{B}_{t+1}\right\}$.

The average goods price is the ratio of expenditures to the quantity of goods the worker buys. Marginal prices have the following interpretation: If at t the worker spends on chained goods, the reduction in her debt is financed with a reduction in chained expenditures. Otherwise, if she does not spend on chained goods, a reduction in her debt is financed with a reduction in spot goods. Since the price of chained goods is q_t and the price of spot goods is 1, the worker sacrifices $1/\tilde{q}^E_t$ units of consumption per unit of debt reduction. Likewise, if she spends spot at $t+1$, any past savings can translate into spot expenditure at $t+1$. Otherwise, if she only spends in chained goods at $t+1$, any past savings translate into chained expenditures. Hence, the worker can buy $1/\tilde{q}^B_{t+1}$ additional goods by reducing her debt at t on the margin. Marginal inflation is the ratio of marginal prices, a definition that enters the following generalized Euler equation.

Proposition 4. (Workers's First-Order Condition): *Fix a sequence* $\left\{\tilde{B}_t, R_{t+1}, q_t\right\}_{t\geq 0}$ *such that* $\tilde{B}_t$ *is an increasing and* $\beta R_{t+1} \leq 1$. *Then, any solution* $\{B_{t+1}\}_{t\geq 0}$ *to the worker's problem satisfies the following generalized Euler equation:*

$$\frac{E^w_{t+1}}{E^w_t}\frac{Q_t}{Q_{t+1}} = \beta \frac{R_{t+1}}{\Pi_{t+1}(B_{t+1})} \quad \text{if} \quad B_{t+1} \neq B^\star_{t+1} \tag{13}$$

and

$$\beta q_t R_{t+1} \geq \frac{E^w_{t+1}}{E^w_t} \geq \beta R_{t+1} \quad \text{if} \quad B_{t+1} = B^\star_{t+1}.$$

This Euler equation is unconventional, but the interpretation is standard. The left-hand side is the ratio of marginal rates of substitution between t and $t+1$ consumption—expressed in expenditures over average prices.[13] The right-hand side captures the relation between discounting and rate of return, βR_{t+1}; the novelty is that it is deflated by marginal inflation. Marginal inflation enters the expression because this is the relevant ratio of prices that deliver marginal utilities at t and $t \dot{+} 1$, respectively $\tilde{q}^E_t$ and $\tilde{q}^B_{t+1}$. This generalized Euler equation holds exactly except at the discontinuity point $B^\star_{t+1}$. The inequalities that are satisfied when $B_{t+1} = B^\star_{t+1}$ correspond to a sub-differential optimality condition: the condition says that increasing debt is optimal to the left of $B^\star_{t+1}$,

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[13]The ratio of expenditures to average prices, is the ratio of marginal utilities under log preferences.

but not to its right.[14]

Importantly, the Euler equation is a necessary condition for optimality, but not sufficient condition. The reason is that the objective function in the worker's problem, (12), is not concave in B'. This introduces further challenges. In particular, multiple (finite) sequences of B_{t+1} indexed by an initial choice of B_1, may satisfy the Euler equation. This property leads to the possibility of hysteresis in general equilibrium. Before explaining this, I present a characterization of the worker's problem in partial equilibrium.

Proposition 5. (Stationary Worker Problem): *Fix* $\tilde{B}_t = B_{ss}$ *and* $R_t = \beta^{-1}$ *and let* $\tilde{B}_{ss} > 1$. *Let* $B_{ss}^\star \equiv B^\star\left(1/\beta, \tilde{B}_{ss}\right)$. *Then, a solution to the worker's problem satisfies:*

I. *If* $B_0 \leq B_{ss}^\star$, *then* $B_t = B_0 \ \forall t$.

II. *There exists a threshold* $B^h > B_{ss}$ *such that:*

$\diamond$ *If* $B_0 < B^h$, *then* $B_t \to B_{ss}^\star$ *in finite time*

$\diamond$ *If* $B_0 > B^h$, *then* $B_t = B_0 \ \forall t$.

The threshold B^h *and the convergent sequence is given in the proof.*

The proposition characterizes the dynamics of B_t: a sharper characterization of convergence in terms of convergence times is provided in its proof. The proposition showcases the domains of attraction toward steady-state solutions. To illustrate the proposition, Figure 3 plots a numerical solution to the worker's Bellman equation and its policy functions when $R_{t+1} = \beta^{-1}$ and $\tilde{B}_t = \tilde{B}_{ss} > 0$. I also overlay the value functions corresponding when $\tilde{B}_t = 0$ and when $\tilde{B}_t = \infty$. I denote these value functions by $\underline{V}$ and $\bar{V}$ respectively.

For debt levels below $B^\star$, V lies on top of $\bar{V}$. When $B < B^\star$, the worker can consume the annuity of his human capital minus the interest on his debt, entirely on spot consumption. This possibility yields the same value, $\bar{V}$, as if the SBL is never binding, $\tilde{B}_t = \infty$. When debt is above a threshold B^h, the value function lies on top of $\underline{V}$. When

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[14]There is also a discontinuity in the Euler equation at $B_{t+1} = \tilde{B}_{t+1}$, but the corresponding sub-differential does not yield an optimality condition.

$B > B^h$, the worker can consume the annuity of his human capital minus the interest on his debt, but in this case, entirely on chained consumption, paying an effective price $q > 1$. When $B > B^h$, even though the worker could potentially save to enjoy better prices in the future, she has no incentives to save. She has no incentives to save because the benefits of deleveraging come too far in the future and require an excessive sacrifice of current consumption. In that case, her value function is $\underline{V}$, as if she could never access short-term funding. For moderate debt levels, when $B \in (B_{ss}^{\star}, B^h]$, the worker will delever until her debt falls below $B^{\star}$. All in all, saving away from financial constraints happens if the SBL constraint is not too tight. This property leads to the possibility of hysteresis.

(a) Value function

(b) Debt Policy
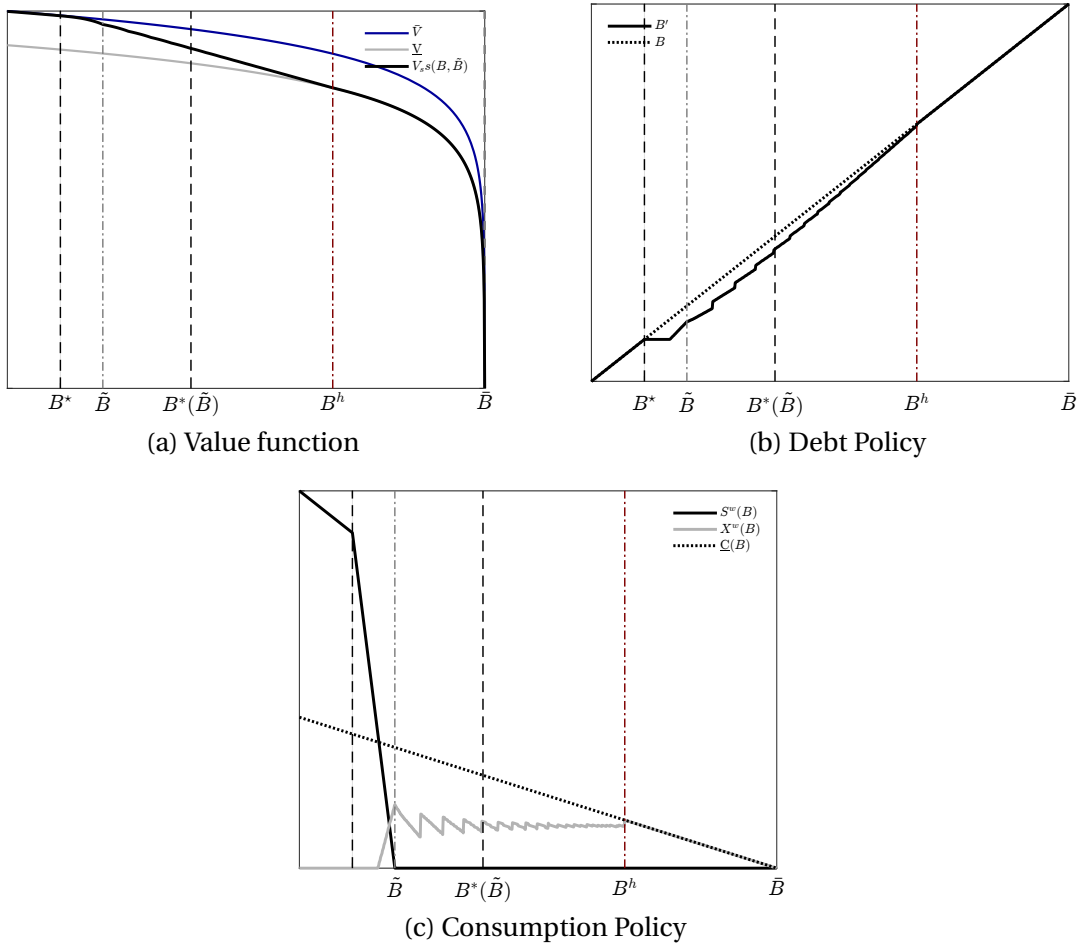
(c) Consumption Policy

Figure 3: Bellman Equation and Policy Functions: Worker Problem

Note: Figures are calculated using value function iteration: $\beta = 0.8$, $q = 1.75$ and $\tilde{B} = 0.4 \cdot \bar{B}$.

Equilibrium Dynamics: From Sequential to a Functional Representation. I now analyze the equilibrium dynamics of B_t. The equilibrium is recursive with state variable $B_t \times \tilde{B}_t \times \tilde{B}_{t+1} \in [0, \bar{B}]^3$. A recursive representation requires that we have a relationship between a sequence and a recursive formulation: a variable m_t, can be expressed as an equilibrium function, $m : [0, \bar{B}]^3 \to \mathbb{R}_+$ such that $m_t = m\left(B_t, \tilde{B}_t, \tilde{B}_{t+1}\right)$. Hence, from now I use m to represent the function that maps the state into its equilibrium value m_t. I also adopt the convention of using m' to refer to m_{t+1}.

Using the recursive formulation, from (9), (7), and (10) we have:

$$C^s(B) = (1 - \beta) B, \quad E^w(B) = 1 - (1 - \beta) B, \tag{14}$$

and

$$S^w\left(B, \tilde{B}\right) = \min\left\{\bar{S}\left(B, \tilde{B}\right), E^w(B)\right\}.$$

Several other equilibrium functions are deduced from these functions directly: $\mu = E^w - S^w$, $q \equiv \mathcal{A}^{-1}(\mu)$, $X^w \equiv \mu/q$, $Q \equiv E^w/C^w$, and, finally, $C^w \equiv S^w + X^w$.

The only endogenous argument of the state is B_t. Thus, we need a map $\mathcal{B}$ from the current state to its future value, $B' = \mathcal{B}\left(B, \tilde{B}, \tilde{B}'\right)$. If we obtain that map, the equilibrium rate will satisfy $R_{t+1} = \mathcal{R}\left(B_t, \tilde{B}_t, \tilde{B}_{t+1}\right)$ where $\mathcal{R}\left(B, \tilde{B}, \tilde{B}'\right) \equiv \beta^{-1}\mathcal{B}\left(B, \tilde{B}, \tilde{B}'\right)/B$, following the saver's optimal rule. With that equilibrium function, we can define the threshold function $B^\star\left(B, \tilde{B}, \tilde{B}'\right) \equiv \mathcal{R}\left(B, \tilde{B}, \tilde{B}'\right) \cdot \left(\tilde{B} - 1\right)$. I solve for $\mathcal{B}$ below, after defining a functional representation for the equilibrium prices:

$$\Pi\left(B'; B, \tilde{B}, \tilde{B}'\right) \equiv \tilde{q}^B\left(B'; \tilde{B}'\right)/\tilde{q}^E\left(B'; B, \tilde{B}, \tilde{B}'\right),$$

where

$$\tilde{q}^E\left(B'; B, \tilde{B}, \tilde{B}'\right) \equiv 1 + \left(q\left(B, \tilde{B}\right) - 1\right) \cdot \mathbb{I}_{\left[B' \geq B^\star\left(B, \tilde{B}, \tilde{B}'\right)\right]}$$

and

$$\tilde{q}^B\left(B'; \tilde{B}'\right) \equiv 1 + \left(q\left(B', \tilde{B}'\right) - 1\right) \cdot \mathbb{I}_{\left[B' > \tilde{B}'\right]}.$$

To find $\mathcal{B}$, I combine the worker and saver Euler equations, (13) and (9). I then substitute out R_{t+1} and use the equilibrium worker expenditures (14), to obtain an equilib-

rium condition B':

$$\underbrace{\frac{B}{1-(1-\beta)\,B}\cdot Q\left(B,\tilde{B}\right)}_{\equiv\mathcal{E}\left(B;\tilde{B}\right)} = \underbrace{\frac{B'}{1-(1-\beta)\,B'}\frac{Q\left(B',\tilde{B}'\right)}{\Pi\left(B';B,\tilde{B},\tilde{B}'\right)}}_{\equiv\mathcal{E}'\left(B';B,\tilde{B},\tilde{B}'\right)}. \tag{15}$$

The left and right hand sides define functions equations $\mathcal{E}\left(B';B\right)$ and $\mathcal{E}'\left(B';B,\tilde{B},\tilde{B}'\right)$. At equality, these functions define the law of motion of debt. Because Q is discontinuous, there may be multiple solutions B'. The following result tells us that in a transition, the lowest value is valid.

Proposition 6. (Equilibrium Rates and Expenditures): *Consider a weakly monotone increasing sequence of spot borrowing lines* $\tilde{B}_t \to \tilde{B}_{ss}$. *For any* $B_0 < B^h\left(\tilde{B}_{ss}\right)$, *if an equilibrium exist, then* $B_{t+1} = \mathcal{B}\left(B_t, \tilde{B}_t, \tilde{B}_{t+1}\right)$ *where*

$$\mathcal{B}\left(B,\tilde{B},\tilde{B}'\right) = \max\left\{B^\star\left(\tilde{B}\right), \ \arg\min_{B'}\left\{\mathcal{E}\left(B';B,\tilde{B},\tilde{B}'\right) = \mathcal{E}'\left(B';B,\tilde{B},\tilde{B}'\right)\right\}\right\}.$$

Proposition 6 is key to describing the dynamics.[15]

Steady States. I use the subscript ss to denote steady states. In principle, workers could make both spot and chained expenditures in steady state. However, from the saver's problem we know that in steady state, $R_{ss} = \beta^{-1}$. To be consistent with the worker's law of motion for debt, the worker's marginal inflation must also equal one. Hence, in a steady state, the worker makes either only spot expenditures or only chained expenditures. I define an **undisrupted steady state** as a steady state with only spot expenditures, and $\mathcal{Y}_{ss} = 1$. In turn, I define an **disrupted steady state** as a steady state where workers only make chained expenditures, and $\mathcal{Y}_{ss} < 1$. The following corollary presents a condition that guarantees that the economy is in an undisrupted steady state.

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[15]Implicitly, the proposition yields an algorithm to compute equilibria. Starting from, B_0, the sequence of debt generated in equilibrium is given by $\mathcal{B}\left(B,\tilde{B},\tilde{B}'\right)$, which is the smallest solution B' to the equation $\mathcal{E}\left(B';B,\tilde{B},\tilde{B}'\right) = \mathcal{E}'\left(B';B,\tilde{B},\tilde{B}'\right)$. For each B, we obtain B' and update the state accordingly.

Corollary 1. *Fix $\tilde{B}_t = \tilde{B}_{ss}$. For any $\tilde{B}_{ss} > 0$, the economy is in an undisrupted steady state for any $B_t \leq B^*_{ss}$.*

If $B_t \leq B^*_{ss}$ the worker can borrow (intra-period) more than his current income net of interests. If that is is the case, the economy is in an undisrupted steady state.[16]

Convergence Toward Undisrupted Steady States. Next, I describe the domain of attraction toward undisrupted steady states. The domain of attraction toward undisrupted steady states is the region for which the equation $\mathcal{E}(B'; B) = \mathcal{E}'\left(B'; B, \tilde{B}, \tilde{B}'\right)$ has a solution where $B' < B$. This region has an upper bound:

$$B^*\left(\tilde{B}\right) \equiv \tilde{B}/\left(C^w\left(\tilde{B}, \tilde{B}\right) + C^s\left(\tilde{B}\right)\right) \geq \tilde{B}.$$

This inequality follows from total consumption being less than total expenditures. We have the following.

Corollary 2. *Let $\tilde{B}_t = \tilde{B}_{ss}$. For any $B_0 < B^*\left(\tilde{B}_{ss}\right)$, $B_{t+1} < B_t$ if $B_t \in \left(B^\star(\beta^{-1}, \tilde{B}_{ss}), \ B^*(\tilde{B}_{ss})\right)$ and $B_{t+1} = B_t$ if $B_t \leq B^\star(\beta^{-1}, \tilde{B}_{ss}, \tilde{B}_{ss})$. If, in addition, $\tilde{B}_{ss} > 1$ the economy converges toward an undisrupted steady state in finite time (and approaches zero debt if $\tilde{B}_{ss} \leq 1$).*

Corollary 2 describes the domain of attraction toward undisrupted steady states. If the worker holds debt between $\tilde{B}$ and $B^*\left(\tilde{B}\right)$ she repays her debts.[17] The deleverage will continue until she reaches a steady state debt once B^*_{ss}. An implication is that payments-chain crises are only temporary in the domain of attraction toward undisrupted steady states.

Figure 4 describes a transition toward an undisrupted steady state. The left panel of the figure plots different debt levels in the x-axis, holding $\tilde{B}$ fixed. The solid blue and dashed gray curves plot the functions $\mathcal{E}$ and $\mathcal{E}'$ correspondingly. The arrows in the figure trace the path of debt generated in equilibrium, following the sequence of solutions to B' implicit in Proposition 6. For initial conditions where $B_0 > B^*_{ss}$, the economy fails to converge. In that region, the only solution to $\mathcal{E} = \mathcal{E}'$ happens at $B' = B$. The middle panel plots the equilibrium interest rate, $R\left(B, \tilde{B}, \tilde{B}'\right)$. The right panel highlights a

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[16]Observe that if $\tilde{B}_{ss} \leq 1$, then $B^\star\left(\beta^{-1}, \tilde{B}_{ss}\right) > 0$ in which case, no undisrupted steady state exists.

[17]Indeed, they will delever at the rate $\mathcal{R} < \beta^{-1}$ consistent with the condition in Proposition 4. In this region, equation (15) may have two solutions, but only the lowest value of B' is optimal.

gray area in the phase diagram, this is the area of hysteresis where a transition to the efficient steady state fails.

Proposition 6 and Corollary 2 help us interpret transitions when the exogenous borrowing limit $\left\{\tilde{B}_t\right\}$ is transitions toward a steady state. Indeed, we can interpret the equilibrium path as the response to a credit crunch event. We know that as long as $B_0 \leq B_{ss}^*$ and $\tilde{B}_{ss} > 1$, the economy converges to an undisrupted steady state. Figure 5 presents an example of a transition. The figure illustrates that there are different phases in a transition. In its extreme phase, at the beginning of the transition, TFP remains depressed at its lowest value because workers only spend by making chained orders and debt remains constant. In the smoothing phase, workers anticipate that they will be able to spend on spot goods in the subsequent period. To smooth consumption, they increase current chained expenditures taking in more debt. This jump in debt is followed by a recovery phase, where workers trade off consumption smoothing and against paying off debt to increase their credit lines and spend spot. The repayment phase corresponds to the last period with positive chained expenditures. Eventually, the economy converges to an efficient steady state with less debt than at the start of the transition. This pattern holds generically.

Debt-Overhang and Hysteresis of payments-chain Crises. I now study hysteresis provoked by debt overhang.

Corollary 3. (Hysteresis): *Let $\tilde{B}_t = \tilde{B}_{ss}$. For $B_t \geq B^h\left(\tilde{B}_{ss}\right)$, the economy is permanently in a disrupted steady state.*

Hysteresis occurs when debt is so large that the consumption sacrifice does not compensate the benefits deleveraging. Recall from Figure 3 that for sufficiently high debt levels, the worker's value function yields the same value as if $\tilde{B}_t = 0$ forever. When these value functions overlap, the economy experiences debt overhang; she prefers to stay put. The value functions are presented for the case where the rate is $R_{t+1} = \beta^{-1}$. Of course, in general equilibrium, the rate is endogenous. Figure 4 depicts the region of hysteresis in the gray area, and shows that the hysteresis region, equilibrium rates are indeed, $R_{t+1} = \beta^{-1}$.[18]

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[18]For that rate, the only solution to the equilibrium condition $\mathcal{E} = \mathcal{E}'$ happens at $B' = B$.

(a) Transition Plot

(b) Equilibrium Rates

(c) Hysteresis Region
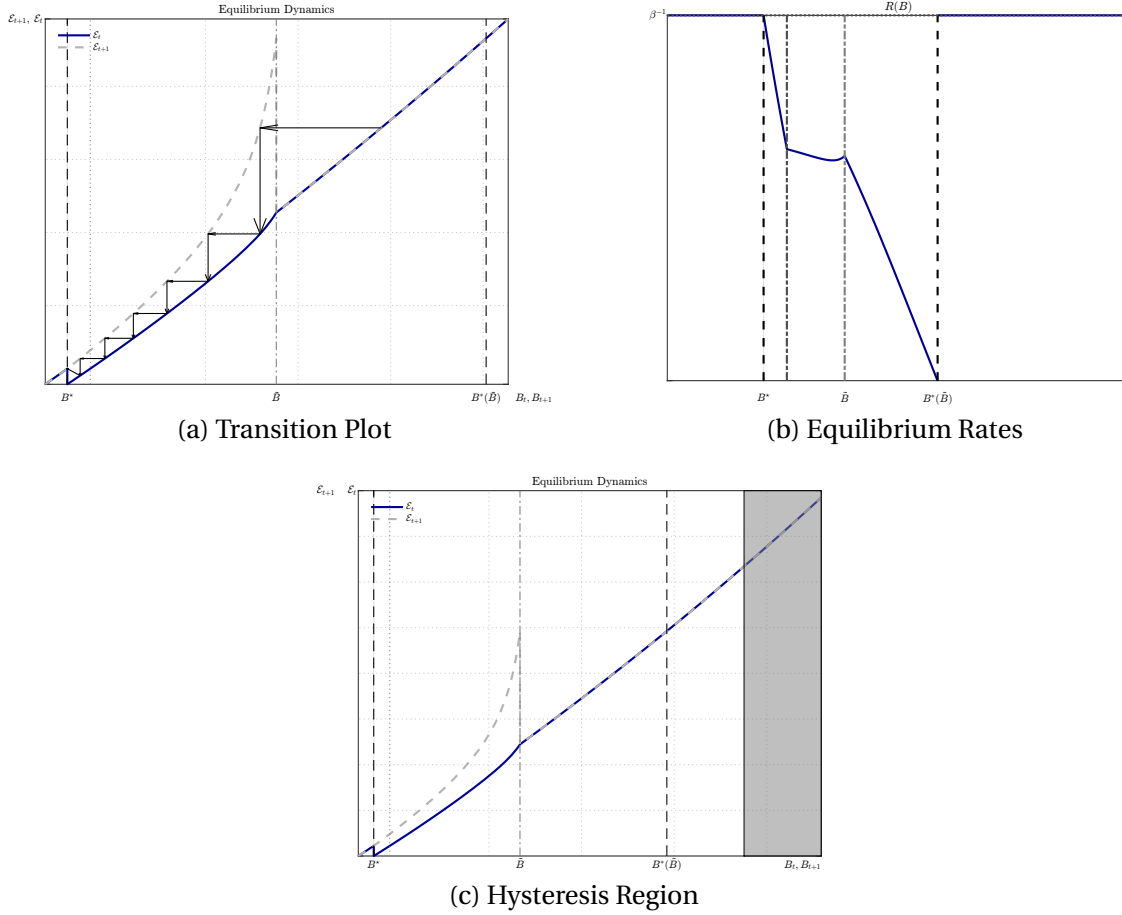
Figure 4: Equilibrium Phase Diagram and Hysteresis Region

Note: Figures are calculated using value function iteration: $\beta = 0.95$, $\delta = 0.9$, and $\tilde{B} = 0.2 \cdot \bar{B}$. Panels (a) and (c) are the phase diagram of B_t constructed by using $\mathcal{E}\left(B'; B\right)$ and $\mathcal{E}'\left(B'; B, \tilde{B}, \tilde{B}'\right)$. Panel (b) plots $R\left(B, \tilde{B}, \tilde{B}'\right)$ in the range of values $[B^\star - 1, B^\star + 2]$. The shaded are in Panel (c) corresponds to the hysteresis region.

If debt starts in hysteresis region, it never falls. Excessive debt hampers the ability to make payments quickly, and the economy ends up in a permanent state of under-capacity production. With deterministic dynamics, hysteresis is a transient state. In a more general setting, a hysteresis-like region where agents don't delever, may be reached via shocks that provoke excess optimism or low discount factors. This form of debt overhang rationalizes several verbal descriptions of the Japanese lost decades or the aftermath of the Euro sovereign debt crisis.

Finally, I note that the domain of attraction of undisrupted steady states and hysteresis are disconnected. This implies that when debt falls between B^*_{ss} and B^h, a symmetric competitive equilibrium does not exist. Exploring multiplicity of equilibria or
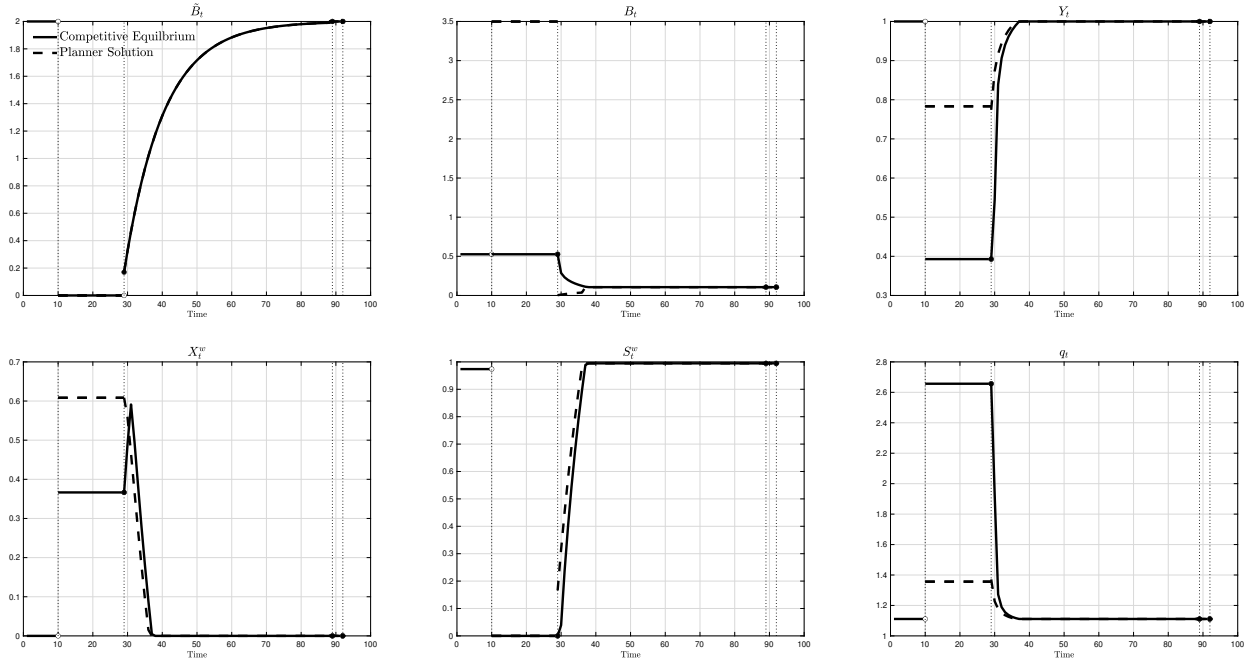
Figure 5: Efficient and Competitive Transition after a Smooth Credit Crunch.

Note: This figure reports a numerical example of a credit crunch episode. Figures are calculated using value function iteration: $\beta = 0.95$, $\delta = 0.9$, and $\tilde{B}_{ss} = 0.1 \cdot B^{\star}$. The crunch happens at period t=10. For $t \in \{11, 29\}$, $\tilde{B}_t = 0$. Then $\tilde{B}_t$ returns to steady state according to an AR(1) process with coefficient 0.9. The transition ends at t=90 when $\tilde{B}_t = \tilde{B}_{ss}$.

solutions with asymmetric behavior is left for future work.[19]

Discussion: Interpretation of Financial Constraints. Saver spot payments represent uses of internal liquid funds, i.e. deposits. The short-term funding limit, $\max\left\{\tilde{B}_t - B_t, 0\right\}$, represent "credit lines", i.e. credit card limits, overdraft facilities, or supply chain finance facilities. Thus, worker spot payments are uses of those lines.[20] In turn, chained payments can be thought of as outstanding account receivables against goods to be delivered. In practice, many account receivables generate from contracts in which the product is not delivered until a fraction of the payment is anticipated. Thus, they rep-

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[19]Note that in this middle region, at an individual level, workers would want to delever as seen in Figure 3. In general equilibrium, this would imply a rate below β^{-1}, for which there are no solutions to equation (15), as shown in Figure 4. The economy may possibly feature sunspot equilibria; a situation I do not consider in this paper.

[20]Other examples of facilities are Business Credit Lines, Standby Letters of Credit and Supplier Finance Programs. Under a Supplier Finance Program, the buyer wants to pay later, whereas suppliers request cash. Supplier Finance enables suppliers to be paid by banks against the receivables. Descriptions of these programs are offered by some of the largest financial institutions: J.P. Morgan Supplier Finance Facility or Citibank Supplier Finance Facility.

resent production agreements.

The SBL,$\tilde{S}$, is different from the debt limit, $\bar{S}$, the hard debt limits that is most prevalent in macro finance models. The $\tilde{B}$ limit applies only to short-term funding, but the worker has the option to roll over its repayment until the debt limit is reached. In the section, we studied contractions in the SBL, while keeping the debt limit constant. There is an economic motivation behind this modeling of credit crunch events: if credit is intermediate by banks, if a bank wants to cut back on credit, it may be convenient to tighten the SBL, but not necessarily to force workers to repay debt principals. If loan repayment is suddenly forced, it can trigger costly defaults.

Finally, it is worth discussing the misallocation of funding. Due to the big family assumption, the worker will receive labor income flows while there are still pending chained orders. This happens because the worker may receive payment on spot orders before it receives payments chained to chained expenditures. The big family assumption is present to avoid a distribution of ex-post outcomes while keeping the simplicity of the TFP function derived earlier. Chained orders would still induce delays even if the worker pool funds from different orders.[21] The option to pool funds will change the exact functional form of the TFP function but not it's essential features.

4. Policy Implications of Payments-Chain Crises

4.1 Constrained Inefficiency

In Section 3, we expressed the chained expenditure ratio as a function of debt levels. In Section 2 we found that higher chained expenditure ratios lead to delays which provoke declines in measured TFP. In part, those losses result from the poor organization of payments chains, given a level of chained expenditures. Ideally, a planner would reorganize payments so that each chain is of equal length, but governments do not have that power. What governments can do is influence spending decisions. This sec-

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[21]For the case where chained orders can be paid with any incoming payment, the chain length distribution can again be derived analytically. In that case, for $N^x > N^s$, the minimal chain length would be zero, the maximal chain length $\lceil N^x/N^s \rceil$, and the probability distribution uniform among chains with $n \leq \lfloor N^x/N^s \rfloor$. For $N^x \leq N^s$, the minimal chain is zero with probability $N^s - N^x$ and the maximal chain length 1.

tion studies such policies. Traffic regulation provides a useful analogy: governments cannot assign drivers into different lanes, but they can tax vehicles. The spirit of the exercise is not normative but, rather, to clarify the sources of constrained inefficiency.

Ideal Pareto Weights.   To study constrained inefficiency, consider a transition in a competitive equilibrium that reaches an undisrupted steady-state debt level B_{ss} starting from a debt level B_0. I study a Ramsey planner problem with Pareto weights on workers, θ, which delivers the same steady-state debt level as the competitive equilibrium:

$$\frac{1 - (1 - \beta) B_{ss}}{(1 - \beta) B_{ss}} = \frac{\theta}{1 - \theta}. \tag{16}$$

Since the economy is efficient in an undisrupted steady state, any difference between the transition path of debt in a planner and competitive equilibrium solutions uncover a constrained inefficiency only during transitions.

The Ramsey Problem.   Consider a sequence of debt taxes $\left\{\tau_t^k\right\}$, labor taxes $\left\{\tau_t^\ell\right\}$, and expenditure taxes $\left\{\tau_t^c\right\}$. The Ramsey Problem is:

Problem 4. (Ramsey Problem): *Given B_0 and $\left\{\tilde{B}_t\right\}$:*

$$\max_{\left\{\tau_{t+1}^k, \tau_t^c, \tau_{t+1}^\ell\right\}_{t \geq 0}} \sum_{t \geq 0} \beta^t \left[(1 - \theta) \log\left(C_t^s\right) + \theta \log\left(C_t^w\right)\right],$$

subject to the (modified) saver budget constraint and optimality:

$$\left(1 + \tau_{t+1}^k\right) R_{t+1}^{-1} B_{t+1} + \left(1 + \tau_t^c\right) C_t^s = B_t, \ \forall t \geq 0,$$

to the (modified) worker budget constraint and optimality:

$$B_t + \left(1 + \tau_t^c\right) E_t^w = R_{t+1}^{-1} B_{t+1} + 1 - \tau_t^\ell, \ \forall t \geq 0,$$

to the government's budget constraint:

$$\tau_{t+1}^k R_{t+1}^{-1} B_{t+1} + \tau_t^c \left(C_t^s + E_t^w\right) + \tau_{t+1}^\ell = 0, \ \forall t \geq 0,$$

and the structure of transactions: (i) optimal expenditures (10-11), (ii) total consumption (2), (iii) the optimality conditions of the worker and saver problems, (iv) the payments constraints (8), and (v) the shadow price, $q_t = \mathcal{A}\left(\mu_t\right)^{-1}$.

The Ramsey planner distorts the competitive equilibrium by using credit, labor, and expenditure taxes. This planner cannot distinguish between the two expenditure forms, but takes into account the agents' optimal behavior, their constraints, market clearing, respects the transactions technology, and satisfies budget balance.

The Primal Problem.   I solve the Ramsey Problem by solving an equivalent Primal Problem.

Proposition 7. (Solution to Ramsey): *The allocations induced by a solution to the Ramsey Problem are the same allocations as the solution to the following problem:*

Problem 5. (Primal Planner Problem):*Taking $\left\{\tilde{B}_t\right\}$ as given:*

$$\max_{\{B_t\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \mathcal{P}\left(B_t, \tilde{B}_t\right)$$

where
$$\mathcal{P}\left(B, \tilde{B}\right) \equiv (1-\theta)\log\left(C^s\left(B\right)\right) + \theta\log\left(S^w\left(B, \tilde{B}\right) + X^w\left(B, \tilde{B}\right)\right). \qquad (17)$$

Let $\left\{B_{t+1}^p\right\}_{t\geq 0}$ be a solution to the Primal Planner Problem. The solution to the Ramsey problem can be implemented setting $(1 + \tau_0^c) = B_0^p/B_0$ and a sequence of debt taxes (a formula is given in the proof).

The proposition asserts that the solution to the Ramsey Problem can obtained from the solution of a Primal Problem where the planner directly chooses the sequence of debt. This relation follows because the constraint set in the Primal Problem includes the constraints of the Ramsey Problem.[22] Hence, if a solution to the primal can be implemented with taxes, it solves the Ramsey Problem. The proposition shows that this is the case.

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[22]This is immediate since market clearing in the asset market and the budget balance, implies, by Walras's law, that the resource constraint holds. In the implementation, capital taxes and a period-zero expenditure tax are required by the Ramsey planner to distort the evolution of debt and replicate the solution to the Primal Problem.

The implementation of the Primal Problem is possible because the Ramsey planner can set capital taxes to induce a desired path of debt $\{B_t\}$. Controlling B_t is key to the implementation. A key property of the environment is that once B_t is known, the time t allocation is determined: Current B_t determines the savers' expenditures through the constant expenditure rule of log utility. Since worker income is always one, and aggregate income equals aggregate expenditures, worker expenditures are given once we know saver expenditures. This is why the objective in the Primal Problem is separable in time (17).

Time separability also implies that expenditure taxes are redundant after $t = 0$. All that the Ramsey planner needs is a sequence of capital taxes to control the path of debt and the labor tax to redistribute the capital tax receipts toward the worker. The planner only needs expenditure taxes at $t = 0$ because debt is predetermined at $t = 0$. Because the Ramsey planner can implement a time-separable primal planner problem, an immediate lemma is that the optimal debt level at time t has a closed form that only depends on the SBL level, $\tilde{B}$, at that period:

Lemma 2. (Static Primal Problem): *The solution to the Primal Planner Problem $B_t = B^p\left(\tilde{B}_t\right)$ where B^p is the solution to the following static problem:*

$$\mathcal{P}^o\left(\tilde{B}\right) = \max_{B \in [0, \tilde{B}]} \mathcal{P}\left(B, \tilde{B}\right).$$

The solution to this static problem is key to understand the inefficiencies of this environment.

Proposition 8. (Solution of the Primal Problem): *The solution to the static problem B^p has the following property:*

I. Efficient Allocation. *For $\tilde{B} \geq \frac{1-\theta\beta}{1-\beta}$, the planner's problem is unconstrained: $\mathcal{A} = 1$ and expenditures are is in steady state (16).*

II. Inefficient Allocations. *For $\tilde{B} < \frac{1-\theta\beta}{1-\beta}$, the planner constraints bind. The planner may or may not distort TFP and induce more or less debt than in steady state:*

II.a. Inefficient Insurance and Efficient Production. *For $\tilde{B} \in \left[\tilde{B}^i, \frac{1-\theta\beta}{1-\beta}\right)$, the planner induces fewer saver expenditures than in steady state, $B^p = B^\star\left(\tilde{B}\right) < B_{ss}$ but production is efficient $\mathcal{A} = 1$. A marginal decrease in debt at B^p increases efficiency.*

II.b. Inefficient Insurance and Inefficient Production *If $\tilde{B}^s < \tilde{B}^i$, production is inefficient, $\mathcal{A} < 1$.*

II.b.i Inefficient Insurance and Inefficient Production | Conflicting Case. *If $\tilde{B} \in \left[\tilde{B}^s, \tilde{B}^i\right)$, the planner induces fewer saver expenditures than in steady state, $B^p < B^\star\left(\tilde{B}\right) < B_{ss}$. B^p is the unique solution to*

$$\frac{1-(1-\beta)B}{(1-\beta)B} = \frac{\theta}{1-\theta}\frac{Q\left(B,\tilde{B}\right)}{q\left(B,\tilde{B}\right)}\left(\frac{q\left(B,\tilde{B}\right) - \beta\left(1 + \epsilon_\mu^{\mathcal{A}}\left(\mu\left(B,\tilde{B}\right)\right)\right)}{1-\beta}\right). \quad (18)$$

In this interval, a marginal decrease in debt at B^p increases efficiency.

II.b.ii Inefficient Insurance and Inefficient Production | Reinforcing Case. *For $\tilde{B} \in \left[0, \tilde{B}^s\right]$, the planner's problem yields a constant lowest value. The planner induces more saver expenditures than in steady state. B^p is the unique solution to:*

$$\frac{1-(1-\beta)B}{(1-\beta)B} = \frac{\theta}{(1-\theta)}\left(1 + \epsilon_\mu^{\mathcal{A}}(1-(1-\beta)B)\right). \quad (19)$$

In this interval, a marginal decrease in debt at B^p decreases efficiency.

The solution to the Primal Problem reveals a novel insight. Namely, during transitions away from payments-chain crises, debt may be inefficiently high or inefficiently low. The intuition behind this is that, for a given debt level, the planner can increase output by reducing or increasing debt. By reducing debt, the planner distributes wealth toward the worker and frees credit lines inducing greater spot expenditures. Alternatively, by increasing debt, the planner distributes wealth toward the saver also stimulating spot expenditures. Both distributive policies distort social insurance, but increase efficiency by speeding up production.

The analytic expression for B^p reveals how the planner allocates expenditures, balancing productive efficiency against social insurance differently, depending on whether $\tilde{B}$ falls in several intervals. There are four intervals of values of $\tilde{B}$ where the planner's solution is qualitatively different. These actual critical values depend on the threshold points $\frac{1+\theta\beta}{1-\beta}$ and $\left\{\tilde{B}^s, \tilde{B}^i, \right\}$ whose formulas are found in the proof.

The first interval of values is where $\tilde{B}$ is a above a threshold level above which the SBL constraint is not binding. In particular, if $\tilde{B} \geq \frac{1+\theta\beta}{1-\beta}$, the planner sets $B = B_{ss}$, and production is efficient. Furthermore, the ratio of marginal utilities equals the ratio of Pareto weights—efficient social insurance is satisfied condition (16).

If $\tilde{B}$ is below the efficiency threshold, $\frac{1+\theta\beta}{1-\beta}$, the planner distorts either social insurance or productive efficiency. The novelty is that the planner has two ways to increase productive efficiency: the planner can increase efficiency on the margin, either by distributing wealth toward the worker if $B < \tilde{B}$ or by distributing wealth toward the saver if $B > \tilde{B}$. To see this, observe that in the region where $B < \tilde{B}$, any further reduction in debt translates, on the margin, into more spot expenditures by the worker and, thus, increases efficiency. In the region where $B > \tilde{B}$, any increase in debt translates, on the margin, into more spot expenditures by the saver, also increasing efficiency. Because of this ambivalent nature, the planner's objective function $\mathcal{P}\left(B, \tilde{B}\right)$ is not concave in B, leading to bang-bang solutions as $\tilde{B}$ varies.

Figure 6 for describing the economics of the Primal Problem, in the regions away from efficiency. The left panel plots B^p for different values of $\tilde{B}$. In the interval of values where $\tilde{B}$ falls in a second interval, i.e. when $\tilde{B} \in [\tilde{B}^i, \frac{1+\theta\beta}{1-\beta})$, the planner solution induces productive inefficiency, $\mathcal{A} = 1$, but but the ratio of marginal utilities no longer the ratio of Pareto weights, as in (16). To understand why the planner does not distort production, observe that because $B^\star\left(\tilde{B}\right) < B_{ss}$, productive efficiency can only be achieved only by setting $B^p = B^\star\left(\tilde{B}\right) < B_{ss}$. Setting debt at $B^\star\left(\tilde{B}\right)$, which is less than its steady-state value, implies that saver expenditures are less than in steady-state. Thus, in this interval the planner sacrifices social insurance by redistributing wealth toward the worker to guarantee productive efficiency. The planner's solution is at a corner because the derivative of his objective is discontinuous at $B^\star\left(\tilde{B}\right)$. This discontinuity follows from the payments-chain network structure: the property that $\mathcal{A}\left(0\right) < 1$, which

captures that even when the chained expenditure ratio is zero, and individual chained order features a production delay.

When $\tilde{B}$ is in a further tighter and falls in the third interval, $\tilde{B} \in \left[\tilde{B}^s, \tilde{B}^i\right)$, the Planner begins to sacrifice productive inefficiency. To maintain productive efficiency, for those levels of the SBL, the planner would have to redistribute even more wealth to the worker at the expense of saver expenditures. In that region, sacrificing social insurance only is not worth it. Hence, the planner sets debt above its efficient level, thus $B^\star\left(\tilde{B}\right) < B^p < \tilde{B}$. Therefore, in this interval, the planner still redistributes wealth (relative to steady state) toward the agent facing the financial constraint, but the planner does partially sacrifice productive efficiency. The planner balances productive efficiency and social insurance, as given by equation (18).

When $\tilde{B}$ falls below an extreme value, in the fourth interval, $\tilde{B} < \tilde{B}^s$, the nature of the planner's solution changes dramatically. Increasing productive efficiency by redistributing wealth toward the worker requires the planner to set debt below $\tilde{B}$. However, when $\tilde{B} < \tilde{B}^s$, setting debt so low would requires an extremely high sacrifice of saver consumption. Below that threshold, the planner prefers to redistribute wealth away from the worker, the constrained agent. Since the planner sets $B > \tilde{B}$, the planner induces only chained expenditures by the worker. Once the worker only makes chained expenditures, the SBL becomes irrelevant, so the planner chooses a constant debt level in this region. This debt level is higher than the unconstrained ideal debt level B_{ss} and given by Equation (19). Debt is higher than in steady state because the planner understands that a marginal increase in debt increases the wealth and, therefore, spot expenditures of savers. This increases productive efficiency.

The ambivalent nature of the planner's problem is germane to the nature payments-chain crises. In typical models with pecuniary externalities, a planner wants to rebalance wealth toward the financially constrained agents to increase productive efficiency. Thus, typically, social insurance and productive efficiency reinforce each other. Here, for extremely low values of $\tilde{B}$, the planner switches to a policy mix where social insurance and efficiency are in conflict. The middle and right panels of Figure 6 illustrates the change in the planner's strategy: In the middle panel, the planner prefers a value of debt where social insurance and efficiency complement. The right panel shows how

he switches strategy as the SBL is tighter.

(a) Planner debt choice

(b) $\mathcal{P}\left(\cdot, \tilde{B}\right)$ (moderate $\tilde{B}$)

(c) $\mathcal{P}\left(\cdot, \tilde{B}\right)$ (tight $\tilde{B}$)
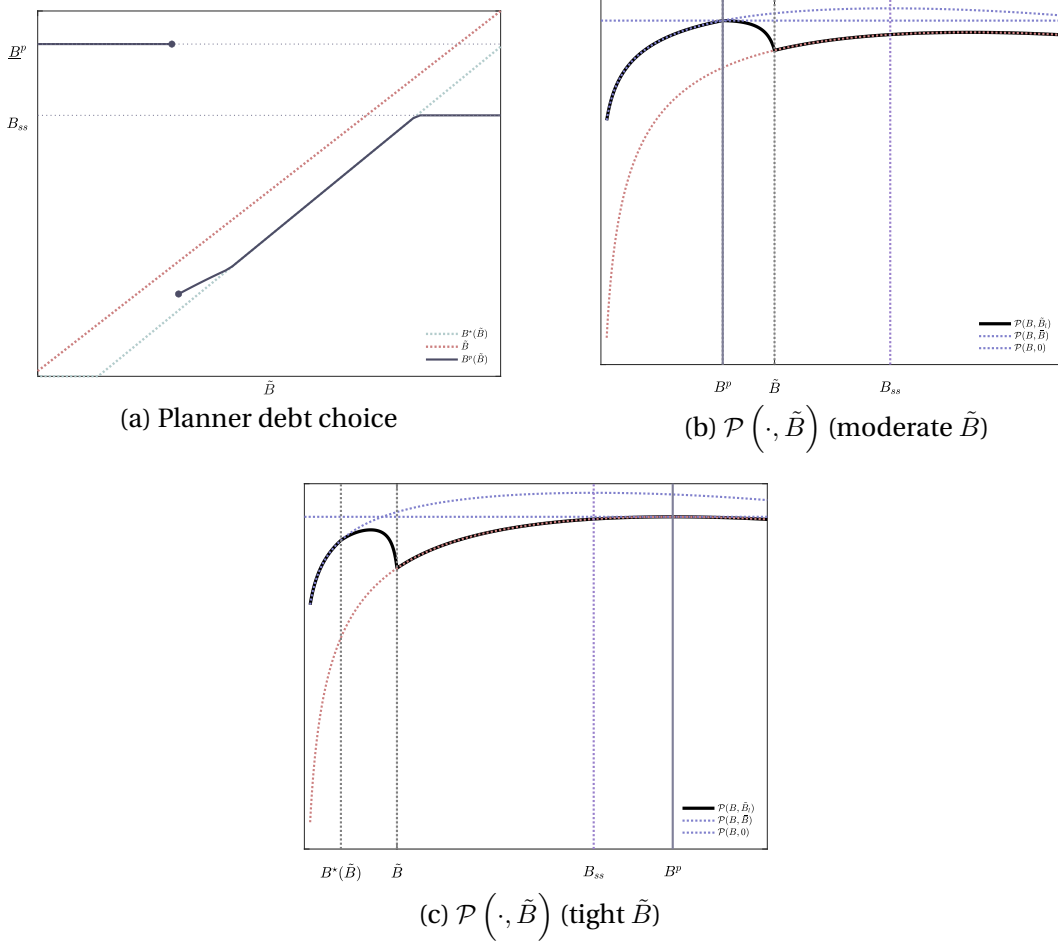
Figure 6: Primal Planner Solution

Note: This figure reports a numerical example of the planner's solution. Figures are calculated using value function iteration: $\beta = 0.95$, $\delta = 0.9$, and $\theta = 0.75$. Panel (a) plots B^p as a function of values of $\tilde{B}$ in the range $[0.1, \theta + 0.1] * B^{star}$. Panel's (b) and (c) plot the objective of the planner for different values of B in the range $[0, 1.8]$. Panel (b) fixes $\tilde{B}$ at 0.15 and Panel (c) at 0.08.

Competitive Equilibrium vs. Efficient Transitions: An illustration. To illustrate how transitions during a payments-chain crisis are inefficient, Figure 5 overlays the planner solution to the competitive equilibrium described above. During the extreme phase of the crisis, the planner redistributes wealth toward savers to induce more spot expenditures. To do so, the planner must subsidize savings. From the outset, the policy seems draconian: the planner taxes the agent suffering most, the worker, to subsidize savers. By making savers wealthier, the planner induces more spot expenditures. The increase in saver wealth increases TFP and, actually, leads to an increase in worker consumption

despite that they are taxed more heavily. The increase in worker consumption comes about through the reduction of the price of chained consumption, which more than offsets the increment in labor taxes. A regressive policy from the outset, increases the welfare of both agents.

During the recovery phase, as $\tilde{B}$ increases and enters the extreme valued interval, the planner reverses the policy. In this region, the planner taxes savings to reduce debt. In doing so, the planner frees some spot-borrowing capacity, and the worker can then make some spot expenditures. As the credit standards are further relaxed, the planner increases debts slowly reaching the efficient steady state.

Though I considered the planner solution along a transition toward an undisrupted steady state, we also know that the planner solution described in Proposition 8 would not allow hysteresis in payments-chain crises.[23] A takeaway is that exiting the hysteresis region, may require policies that seem draconian from the outset.

4.2 Fiscal Policy and the Bocola Effect

Fiscal Policy: when the government pays matters. I now consider government expenditures. I distinguish between spot government expenditures and expenditures chained to future tax receipts. It turns out that the type of government expenditures matters. I call this effect the Bocola effect, because economist Luigi Bocola suggested this distinction.

To formulate the Bocola effect, I treat government expenditures as isomorphic to household expenditures: I label by G^s the spot government expenditures. I assume that the government must also borrow intra-period. In turn, the government can make chained expenditures, G^x. For that, I treat government taxes as income units isomorphic to the labor income of households. For simplicity, I assume that the government raises labor taxes and satisfies a balanced budget at the end of the period. Furthermore, the resources used by both forms of expenditures are wasted. I consider the following problem.

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[23]Indeed, in the hysteresis region, $Q_t/q_t = 1$, so $B_t = B_{t+1}$ is the only solution to the debt accumulation in the competitive equilibrium, (15). In the planner's solution, that equation is altered by the term, $\left(1 - \epsilon_{\mu,t}^q\right) < 1$.

Problem 6. (Government Problem with Expenditures): *Given $B_0 = B_{ss}$ and $\left\{\tilde{B}_t\right\}$:*

$$\mathcal{P}_0^g = \max_{\left\{\tau_t^k, G_t^x, G_t^s\right\}_{t \geq 0}} \sum_{t \geq 0} \beta^t \left[(1-\theta)\log\left(C_t^s\right) + \theta \log\left(C_t^w\right)\right],$$

subject to the saver's optimal behavior (9), a worker budget constraint:

$$B_t + E_t^w = R_{t+1}^{-1} B_{t+1} + 1 - \tau_t^\ell, \ \forall t \geq 0$$

the worker's optimal behavior (13), (10-11), and (2), a government budget-balance constraint with expenditures:

$$\tau_t^\ell = G_t^x + G_t^s, \ \forall t \geq 0,$$

and the ratio of chained expenditures relative to total expenditures:

$$\mu_t = q_t X_t^w + G_t^x \ and \ q_t = \mathcal{A}\left(\mu_t\right)^{-1}.$$

The problem is similar to the Ramsey Problem but this new problem includes government expenditures and excludes credit taxes. Instead of solving the problem, I compute government multipliers near no expenditures. These multipliers transparently showcase how the payment time of government expenditures matters:

Proposition 9. (Infinitesimal Government Multiplies): *Fix $\{G^x, G^s\} = (0,0)$. Consider an unexpected marginal increase in government expenditures of type $g \in \{x, s\}$ at time t. We have that:*

$$\frac{\partial \mathcal{P}_t^g}{\partial G^g} = \underbrace{\frac{\theta}{C^w}}_{marginal \ ind. \ utility} \times \underbrace{\frac{\partial C^w}{\partial G^g}}_{multiplier} \ \ for \ g \in \{x, s\}.$$

The consumption responses are:

$$\frac{\partial C^w}{\partial G^x} = \begin{cases} -1 & B < B^\star\left(\tilde{B}\right) \\ \\ -\mathcal{A}\left(\mu\right) & B > B^\star\left(\tilde{B}\right) \end{cases}, \qquad \frac{\partial C^w}{\partial G^s} = \begin{cases} -1 & B < B^\star\left(\tilde{B}\right) \\ \\ -\mathcal{A}\left(\mu\right)\left(1 + \epsilon_\mu^{\mathcal{A}}\right) & B > B^\star\left(\tilde{B}\right). \end{cases}$$

Finally, the output multipliers are:

$$\frac{\partial \mathcal{Y}}{\partial G^x} = \begin{cases} \mathcal{A}\left(\mu\right)\left(1 + \epsilon_\mu^{\mathcal{A}}\right) - 1 & B < B^\star\left(\tilde{B}\right) \\ \\ 0 & B > B^\star\left(\tilde{B}\right) \end{cases}, \quad \frac{\partial \mathcal{Y}}{\partial G^s} = \begin{cases} 0 & B < B^\star\left(\tilde{B}\right) \\ \\ 1 - \mathcal{A}\left(\mu\right)\left(1 + \epsilon_\mu^{\mathcal{A}}\right) & B > B^\star\left(\tilde{B}\right). \end{cases}$$

When $B \leq B^\star$, production is efficient. For such low levels of debt, Proposition 9 shows that any form of government expenditures is a waste. First, either form of government expenditure reduces worker expenditures one-for-one, leading to a reduction in welfare. In terms of output, if the expenditure is spot, the multiplier is zero as transfers spot expenditures from the private to the public sector. If in turn, government expenditures are chained, the income multiplier is negative because these government expenditures increase productive inefficiency, $\left(1 - \mathcal{A}\left(\mu\right)\left(1 + \epsilon_\mu^{\mathcal{A}}\right)\right) > 0$.

In a payments-chain crisis, when $B > B^\star$, the multipliers behave differently. Chained expenditures are again detrimental for welfare.[24] They carry a zero output multiplier because they transfer an inefficient source of expenditures from the private sector to the public sector. By contrast, if the government spends spot, it provokes a positive externality. This externality is captured by the elasticity $\epsilon_\mu^{\mathcal{A}}$. Spot government expenditures also crowd-out worker chained expenditures, but the income extracted from workers are spent upfront. Ultimately, this reduces the average chain length and increases TFP. In a deep crisis, spot government expenditures may even increase worker consumption. In particular, this occurs when $\epsilon_\mu^{\mathcal{A}} < -1$, a condition that may indeed occurs, as shown in the proof. In terms of the output multiplier of spot expenditures, it is always positive in a payments chain crisis. The multiplier is positive in deep recessions because the government substitutes inefficient private expenditures for efficient public expenditures. Welfare increases precisely when the multiplier is above one.

We learn a new lesson. In a payments-chain crisis, government expenditures can increase welfare but only when paid upfront during deep crises. The argument is different from the classic aggregate-demand stimulus arguments. In this setting, a government that subscribes to the idea that it can stimulate aggregate demand, simply by

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[24]Government expenditures crowd-out worker chained expenditures one-for-one. Divided by $1/q$, this gives us the reduction in chained consumption of chained government expenditures.

spending without considering when it pays for these expenditures, would unwillingly reduce welfare and have no output effects. To produce positive effects, the government must spend by paying for things upfront.

Discussion: Fiscal Transfers, and Ricardian Equivalence. Because the labor is inelastic to labor taxes and saver expenditures are inelastic to capital taxation, the Ramsey planner is equivalent to one that can set transfers.[25] The exercise above therefore shows that Ricardian equivalence fails in this setting. For example, future transfers can have pervasive effects in the midst of a payments-chain crisis. The reason is that it may lead workers to spend more. If workers increase chained expenditures, this may reduce output.

5. Conclusion

The contribution of this paper is to propose a payments interpretation of financial crises. The economic problem here is the inefficient timing of payments. This inefficiency causes production delays and is a coordination failure aggravated by limited funding. These inefficiencies are encoded in $\mathcal{A}$. Whereas the policy recommendations have a similar flavor to those in environments with demand externalities, the paper shows that policies should be directed at accelerating payments.

I made two shortcuts. First, I assume that all transactions are bilateral and for the identical amounts. In practice, payments and production are more complex. Second, I assumed that households produce. In practice, payment chains are more relevant for firms. Developing payments-chain networks with richer transactions and firm production is important to bring realism. Nonetheless, I expect the lessons here to hold in more general settings.

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[25]An equivalence holds as long as transfers cannot be immediately used for spot payments.

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Appendix (Not intended for publication)

Contents

A. Proofs of Section 2

A.1 Proof of Proposition 2

Part 1. Derivation of TFP. In the body of the paper, I showed that for a chain of three orders, of which two are chained, the vector of production is $\{1, \delta, \delta^2\}$. The following induction argument generalizes: If the k-th node initiates production at time $1 - \delta^{k-1}$, the delay from its inspection is $(1 - \delta)\,\delta^{k-1}$, which added to previous delays leads to a transfer of funds only by time $1 - \delta^{k-1} + (1 - \delta)\,\delta^{k-1} = 1 - \delta^k$. This, leaves δ^k time for production to the subsequent unit. Since we computed the delay for $k = 1, 2, \dots$. the productions in a chain of length n are $\{1, \delta, \delta^2, \dots, \delta^n\}$.

It follows that the average output for chained orders, that is excluding the output of the spot order in the chain, in a payments-chain of length n is

$$\bar{y}_n^x = \frac{1}{n} \sum_{m=1}^{n} \delta^m = \frac{\delta}{n} \left(\frac{1 - \delta^n}{1 - \delta} \right).$$

Proof. Recall that a chained order will necessarily fall in a chain with length $n \geq 1$. Thus, the p.m.f of lengths conditional on this event is

$$G^x(n; \mu) = \frac{(1 - \mu)\,\mu^n}{\mu}.$$

Next, we turn to our goal of finding the expected output of a chained order:

$$
\begin{aligned}
\mathbb{E}\left[\bar{y}^x\right] &= \sum_{n=1}^{\infty} \bar{y}_n^x G^x(n; \mu), \\
&= \sum_{n=1}^{\infty} \frac{(1 - \mu)\,\mu^n}{\mu} \cdot \frac{\delta}{n} \left(\frac{1 - \delta^n}{1 - \delta} \right), \\
&= \frac{(1 - \mu)}{\mu} \cdot \frac{\delta}{(1 - \delta)} \cdot \sum_{n=1}^{\infty} \left(\frac{\mu^n}{n} - \frac{(\delta\mu)^n}{n} \right), \\
&= \frac{1 - \mu}{\mu} \cdot \frac{\delta}{1 - \delta} \cdot \ln \left(\frac{1 - \delta\mu}{1 - \mu} \right),
\end{aligned}
$$

The last equality follows from:

$$\sum_{n=1}^{\infty} a^{n-1} = \frac{1}{1-a} \leftrightarrow$$

$$\sum_{n=1}^{\infty} \frac{a^n}{n} = \ln\left(\frac{1}{1-a}\right)$$

for $|a| < 1$, which can be shown by simply taking derivatives to both sides. A simple weak law of large numbers yields the desired result. Recall that we can have spot identities $i \in \mathcal{N}^s$ as unique identifiers for payment chains (spot orders for a one-to-one map with chains). Let $\ell(i)$ be the length of the payment chain that starts with spot order i. Given that for each $i \in \mathcal{N}^s$, $\ell(i) \sim i.i.d.\, G(\mu)$, then

$$\mathcal{A}(\mu; \delta) \equiv \plim_{N \to \infty} \frac{1}{N^s} \sum_{i \in \mathcal{N}^s} \overline{y}_{\ell(i)}^x = \mathbb{E}\left[\overline{y}^x\right].$$

In words, the average output among chained orders converges to $\mathbb{E}\left[\overline{y}^x\right]$ as the network gets larger ($N \to \infty$). $\qquad\square$

Next, we derive expected output—the limit as $N \to \infty$ is implicitly. The fraction of spot orders is $(1 - \mu)$. Production is 1 in their case. The fraction of chained orders is μ, and they produce on average $\mathcal{A}(\mu)$. Thus, total output is:

$$\mathcal{Y}(\mu) = (1 - \mu) + \mu\mathcal{A}(\mu).$$

Next, we obtain the derivative and limits of $\mathcal{Y}(\mu), \mathcal{A}(\mu)$.

Part 2. Limits. We first consider the limit as $\mu \to 0$:

$$\lim_{\mu \to 0} \mathcal{A}(\mu; \delta) = \frac{\delta}{(1-\delta)} \lim_{\mu \to 0} \left(\frac{1}{\mu} - 1\right) \cdot \ln\left(\frac{1-\delta\mu}{1-\mu}\right) = \lim_{\mu \to 0} \frac{\ln\left(\frac{1-\delta\mu}{1-\mu}\right)}{\mu}.$$

The last term is the ratio of two variables that converge to zero. Using L'Hospital's rule:

$$\lim_{\mu \to 0} \frac{\ln\left(\frac{1-\delta\mu}{1-\mu}\right)}{\mu} = \frac{\delta}{(1-\delta)} \frac{\lim_{\mu \to 0}\left(\frac{1}{1-\mu} - \frac{\delta}{1-\delta\mu}\right)}{1} = \delta.$$

where I used

$$\frac{\partial \ln\left(\frac{1-\delta\mu}{1-\mu}\right)}{\partial\mu} = \frac{1-\mu}{1-\delta\mu}\left(\frac{1-\delta\mu}{1-\mu}\right)\left(\frac{1}{1-\mu} - \frac{\delta}{1-\delta\mu}\right) = \left(\frac{1}{1-\mu} - \frac{\delta}{1-\delta\mu}\right).$$

For output, the limit is:

$$\lim_{\mu\to 0}\mathcal{Y}(\mu;\delta) = \lim_{\mu\to 0}(1-\mu)\lim_{\mu\to 0}\left(1 + \frac{\delta}{1-\delta}\ln\left(\frac{1-\delta\mu}{1-\mu}\right)\right) = 1.$$

Next, we consider the limit as $\mu \to 1$:

$$\lim_{\mu\to 1}\mathcal{A}(\mu;\delta) = \frac{\delta}{(1-\delta)}\lim_{\mu\to 1}\left(\frac{1}{\mu} - 1\right)\lim_{\mu\to 1}\ln\left(\frac{1-\delta\mu}{1-\mu}\right).$$

This is the product of numbers that go to 0 and infinity. Using L'Hospital's rule:

$$\lim_{\mu\to 1}\mathcal{A}(\mu;\delta) = \frac{\lim_{\mu\to 1}\left(-\frac{1}{\mu^2}\right)}{\lim_{\mu\to 1}\left(\frac{1}{1-\mu} - \frac{\delta}{1-\delta\mu}\right)} = 0.$$

For output, the limit is:

$$\lim_{\mu\to 1}\mathcal{Y}(\mu) = \lim_{\mu\to 1}(1-\mu) + \lim_{\mu\to 1}\mu\lim_{\mu\to 1}\mathcal{Y}^x(\mu) = 0.$$

Next, we consider the limit as limit as $\delta \to 0$:

$$\lim_{\delta\to 0}\mathcal{A}(\mu;\delta) = \left(\frac{1}{\mu} - 1\right)\lim_{\delta\to 0}\frac{\delta}{(1-\delta)}\cdot\lim_{\delta\to 0}\ln\left(\frac{1-\delta\mu}{1-\mu}\right) = 0.$$

For output,

$$\lim_{\delta\to 0}\mathcal{Y}(\mu) = (1-\mu) + \mu\lim_{\delta\to 0}\mathcal{Y}^x\mathcal{A}(\mu;\delta) = (1-\mu).$$

Finally, we consider the limit as $\delta \to 1$:

$$\lim_{\delta\to 1}\mathcal{A}(\mu;\delta) = \left(\frac{1}{\mu} - 1\right)\lim_{\delta\to 1}\delta\cdot\lim_{\delta\to 1}\frac{1}{(1-\delta)}\cdot\lim_{\delta\to 1}\ln\left(\frac{1-\delta\mu}{1-\mu}\right).$$

This derivative is of the ratio of two numbers that go to zero. Using L'Hospital's rule:

$$\lim_{\delta \to 1} \mathcal{A}\left(\mu; \delta\right) = \left(\frac{1}{\mu} - 1\right) \frac{\lim_{\delta \to 1} \frac{1-\mu}{1-\delta\mu}\left(\frac{-\mu}{1-\mu}\right)}{-1} = \left(\frac{1-\mu}{\mu}\right)\left(\frac{\mu}{1-\mu}\right) = 1.$$

For output,

$$\lim_{\delta \to 1} \mathcal{Y}\left(\mu\right) = \left(1 - \mu\right) + \mu \lim_{\delta \to 1} \mathcal{Y}^x \mathcal{A}\left(\mu; \delta\right) = 1.$$

This concludes the derivation of the limits of interest.

Part 3. Monotonicity. Next, we investigate the derivatives of $\mathcal{A}$ and $\mathcal{Y}$. We can write:

$$\mathcal{A}\left(\mu; \delta\right) = \left(\frac{1}{\mu} - 1\right) \cdot \frac{\delta}{\left(1-\delta\right)} \cdot \ln\left(\frac{1-\delta\mu}{1-\mu}\right).$$

Thus,

$$\mathcal{A}_\mu = \frac{\delta}{\left(1-\delta\right)}\left(\left(-\frac{1}{\mu^2}\right) \cdot \ln\left(\frac{1-\delta\mu}{1-\mu}\right) + \left(\frac{1}{\mu} - 1\right)\left(\frac{-\delta}{1-\delta\mu} + \frac{1}{1-\mu}\right)\right).$$

Factoring out $-1/\mu^2$:

$$\mathcal{A}_\mu = -\frac{\delta}{\left(1-\delta\right)}\frac{1}{\mu^2}\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(1-\mu\right)\left(\frac{-\delta}{1-\delta\mu} + \frac{1}{1-\mu}\right)\right)$$

$$= -\frac{\delta}{\left(1-\delta\right)}\frac{1}{\mu^2}\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right).$$

To show that the derivative is negative, we need to show that the term in the parenthesis is positive. Or likewise that

$$\ln\left(1-\delta\mu\right) - \left(\frac{\mu-\delta\mu}{1-\delta\mu}\right) > \ln\left(1-\mu\right).$$

A concave function $f\left(x\right) \equiv \log\left(1-x\right)$ satisfies:

$$f\left(x\right) + f'\left(x\right)\left|y - x\right| > f\left(y\right).$$

Let $x = \delta\mu$ and $y = \mu$. Because $\{\delta, \mu\} < 1$,

$$|y - x| = \mu - \delta\mu.$$

Thus, we have:

$$\ln\left(1 - \delta\mu\right) + \underbrace{\left(-\frac{1}{1 - \delta\mu}\right)\left(\mu - \delta\mu\right)}_{f'(x)|y-x|} > \ln\left(1 - \mu\right).$$

which proves the desired inequality. Hence, $\mathcal{A}_\mu < 0$ for any $\mu > 0$. At $\mu = 0$, the derivative is zero.

We also obtain that:

$$\mathcal{Y}_\mu = -1 + \mu\mathcal{A}_\mu + \mathcal{A} = -\underbrace{(1 - \mathcal{A})}_{>0} - \frac{\delta}{(1 - \delta)}\frac{1}{\mu}\left(\underbrace{\left(\ln\left(\frac{1 - \delta\mu}{1 - \mu}\right) - \mu\left(\frac{1 - \delta}{1 - \delta\mu}\right)\right)}_{>0}\right) \quad \text{for } \mu > 0.$$

The derivative is also zero at $\mu = 0$.

Part 4. Concavity. Next we perform the convexity analysis. $\mathcal{A}_{\mu\mu}$ is

$$\frac{\delta}{(1 - \delta)}\left[2\frac{1}{\mu^3}\left(\ln\left(\frac{1 - \delta\mu}{1 - \mu}\right) - \mu\left(\frac{1 - \delta}{1 - \delta\mu}\right)\right) - \frac{1}{\mu^2}\left(\frac{1}{1 - \mu} - \frac{\delta}{1 - \delta\mu} - \frac{1 - \delta}{1 - \delta\mu} - \mu\delta\frac{1 - \delta}{(1 - \delta\mu)^2}\right)\right]$$

After some algebraic manipulations, the second term in parenthesis simplifies to:

$$\frac{1}{1 - \mu} - \frac{\delta}{1 - \delta\mu} - \frac{1 - \delta}{1 - \delta\mu} - \mu\delta\frac{1 - \delta}{(1 - \delta\mu)^2} = \mu\frac{(\delta - 1)^2}{(1 - \mu)(1 - \delta\mu)^2}.$$

Thus:

$$\mathcal{A}_{\mu\mu} = \frac{\delta}{(1 - \delta)}\frac{1}{\mu^3}\left[2\ln\left(\frac{1 - \delta\mu}{1 - \mu}\right) - 2\mu\left(\frac{1 - \delta}{1 - \delta\mu}\right) - \frac{\mu^2(1 - \delta)^2}{(1 - \mu)(1 - \delta\mu)^2}\right].$$

We can add the second and third terms to obtain:

$$\mathcal{A}_{\mu\mu} = \frac{\delta}{(1 - \delta)}\frac{1}{\mu^3}\left[\ln\left(\frac{1 - \delta\mu}{1 - \mu}\right)^2 - \mu\left(\frac{1 - \delta}{1 - \delta\mu}\right)\left(\frac{2 - \mu - 3\delta\mu + 2\delta\mu^2}{(1 - \mu)(1 - \delta\mu)}\right)\right].$$

This function is strictly concave if:

$$\ln\left(1 - \delta\mu\right)^2 < \ln\left(1 - \mu\right)^2 + \frac{1}{\left(1 - \mu\right)^2}\left(\frac{\mu\left(1 - \mu\right)\left(1 - \delta\right)}{\left(1 - \delta\mu\right)^2}\left(2 - \mu + 2\delta\mu^2 - 3\delta\mu\right)\right) \quad (20)$$

Let $F\left(x\right) = \ln\left(x\right)$. Set $x_0 = \left(1 - \mu\right)^2$ and $x_1 = \left(1 - \delta\mu\right)^2$ so that

$$x_0 - x_1 = -\mu\left(1 - \delta\right)\left(2 - \mu\left(1 + \delta\right)\right).$$

By strict concavity of $\ln\left(x\right)$ we have

$$\ln\left(1 - \delta\mu\right)^2 < \ln\left(1 - \mu\right)^2 - \frac{1}{\left(1 - \mu\right)^2}\mu\left(1 - \delta\right)\left(2 - \mu\left(1 + \delta\right)\right)$$

so to prove that $\mathcal{A}_{\mu\mu}$ is strictly negative, we need to prove that the right hand side of the expression above is smaller than the condition needed for concavity, condition (20),

$$-\frac{1}{\left(1 - \mu\right)^2}\mu\left(1 - \delta\right)\left(2 - \mu\left(1 + \delta\right)\right) \leq \frac{1}{\left(1 - \mu\right)^2}\left(\frac{\mu\left(1 - \mu\right)\left(1 - \delta\right)}{\left(1 - \delta\mu\right)^2}\left(2 - \mu + 2\delta\mu^2 - 3\delta\mu\right)\right).$$

Cancelling common terms and rearranging, this condition is equivalent to:

$$-\left(1 - \delta\mu\right)^2\left(2 - \mu - \mu\delta\right) \leq \left(1 - \mu\right)\left(2 - \mu + 2\delta\mu^2 - 3\delta\mu\right). \quad (21)$$

The term on the left is negative—and strictly negative for $\delta, \mu < 1$.[26] Hence, the inequality above is verified as long as:

$$2 \geq \alpha\left(\mu, \delta\right) \equiv \mu - 2\delta\mu^2 + 3\delta\mu.$$

Hence, as long as

$$2 \geq \alpha^* = \max_{\{\mu,\delta\}\in[0,1]^2} \alpha\left(\mu, \delta\right)$$

the condition for concavity holds for all $\{\mu, \delta\} \in [0, 1]^2$. We study this max function. Fix any μ. Since

$$\delta\left(3\mu - 2\mu^2\right) \geq 0,$$

---

[26]This follows immediately because μ and δ are fractions.

the maximal value of α is achieved when $\delta = 1$. Hence, the objective is

$$\alpha^* = \max_{\{\mu,\delta\}\in[0,1]^2} \alpha\left(\mu,\delta\right) = \max_{\{\mu\}\in[0,1]} \alpha\left(\mu,1\right) = \max_{\{\mu\}\in[0,1]} 4\mu - 2\mu^2.$$

Maximizing the last expression over μ yields $\mu = 1$ as a solution and $\alpha^* = 2$ as the value. Hence, the inequality holds and guarantees (21). This suffices to prove concavity.

Next, we verify the concavity of total output. We have that $\mathcal{Y}_{\mu\mu}$ is

$$=\mathcal{A}_\mu + \mu\mathcal{A}_{\mu\mu}$$
$$=-\frac{\delta}{(1-\delta)\,\mu^2}\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right) + \frac{\delta}{(1-\delta)\,\mu^2}2\mu\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right)$$
$$\quad-\frac{\mu}{\mu^2}\left(\frac{1}{1-\mu} - \frac{\delta}{1-\delta\mu} - \frac{1-\delta}{1-\delta\mu} - \mu\delta\frac{1-\delta}{(1-\delta\mu)^2}\right)$$
$$=\mathcal{A}_{\mu\mu} - \frac{\delta}{(1-\delta)\,\mu^3}\mu\left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right)\right)$$
$$<0.$$

A.2 Related Results Used Elsewhere

In this section, we derived properties that are used later in the text.

Part 5. Inverse productivity. Now, we study the inverse of productivity. Let

$$q\left(\mu;\delta\right) = \mathcal{A}^{-1}\left(\mu;\delta\right).$$

Clearly, the function has the limits:

$$\lim_{\mu\to0} q\left(\mu;\delta\right) = \delta^{-1} \text{ and } \lim_{\mu\to1} q\left(\mu;\delta\right) = \infty \text{ and } \lim_{\delta\to0} q\left(\mu;\delta\right) = \infty \text{ and } \lim_{\delta\to1} q\left(\mu;\delta\right) = 1.$$

We also have that:

$$q_\mu = -\frac{\mathcal{A}_\mu}{\mathcal{A}^2} > 0.$$

We use the limit of the derivative of this function:

$$q_\mu(\mu) = \frac{\frac{\delta}{(1-\delta)} \frac{1}{\mu^2} \left(\ln\left(\frac{1-\delta\mu}{1-\mu}\right) - \mu\left(\frac{1-\delta}{1-\delta\mu}\right) \right)}{\left(\frac{(1-\mu)}{\mu} \cdot \frac{\delta}{(1-\delta)} \cdot \ln\left(\frac{1-\delta\mu}{1-\mu}\right) \right)^2}$$

Next, we check the convexity of function:

$$q_{\mu\mu} = -\frac{\mathcal{A}_{\mu\mu}}{\mathcal{A}^2} + \frac{\mathcal{A}_\mu}{\mathcal{A}^3} > 0.$$

Hence, q is convex in μ.

Part 6. Elasticity of $\mathcal{A}$. A useful object in later derivations is the elasticity of $\mathcal{A}$. Consider the derivative of

$$\mathcal{A}(\mu)\,\mu.$$

We have:

$$\mathcal{A}'(\mu)\,\mu + \mathcal{A}(\mu) = \mathcal{A}(\mu)\left[1 + \epsilon_\mu^{\mathcal{A}}\right].$$

Recall that,

$$\mathcal{A}(\mu)\,\mu = (1-\mu)\left(\frac{\delta}{1-\delta} \ln\left(\frac{1-\delta\mu}{1-\mu}\right) \right).$$

Hence,

$$
\begin{aligned}
\mathcal{A}'(\mu)\,\mu + \mathcal{A}(\mu) &= \frac{\delta}{1-\delta}\left(-\ln\left(\frac{1-\delta\mu}{1-\mu}\right) + (1-\mu)\left(\frac{-\delta}{1-\delta\mu} + \frac{1}{1-\mu}\right) \right) \\
&= \frac{\delta}{1-\delta}\left(-\ln\left(\frac{1-\delta\mu}{1-\mu}\right) + \left(\frac{1-\delta}{1-\delta\mu}\right) \right).
\end{aligned}
$$

Hence, we obtain that:

$$
\begin{aligned}
\left[1 + \epsilon_\mu^{\mathcal{A}}\right] &= \frac{\frac{\delta}{1-\delta}\left(-\ln\left(\frac{1-\delta\mu}{1-\mu}\right) + \left(\frac{1-\delta}{1-\delta\mu}\right) \right)}{\mathcal{A}(\mu)} \\
&= \frac{\mu}{1-\mu}\left(\frac{\frac{1-\delta}{1-\delta\mu}}{\ln\left(\frac{1-\delta\mu}{1-\mu}\right)} - 1 \right).
\end{aligned}
$$

We are interested in the sign of $1 + \epsilon_\mu^{\mathcal{A}}$ and its limits. We know $\epsilon_\mu^{\mathcal{A}} < 0$. Thus, the sign

of $1 + \epsilon_\mu^{\mathcal{A}}$ is the sign of:

$$\left(\frac{1 - \delta}{1 - \delta\mu} \right) - \ln\left(\frac{1 - \delta\mu}{1 - \mu} \right).$$

The limits of the function that governs the sign are:

$$\lim_{\mu \to 0} \left(\frac{1 - \delta}{1 - \delta\mu} \right) - \ln\left(\frac{1 - \delta\mu}{1 - \mu} \right) = (1 - \delta) > 0,$$

and

$$\lim_{\mu \to 1} = \left(\frac{1 - \delta}{1 - \delta\mu} \right) - \ln\left(\frac{1 - \delta\mu}{1 - \mu} \right) = -\infty.$$

Since the function is continuous in μ, the sign is ambiguous.

Next, to establish monotonicity, notice that

$$
\begin{aligned}
\delta\frac{1 - \delta}{(1 - \delta\mu)^2} - \frac{1 - \delta}{(1 - \delta\mu)(1 - \mu)} &= (1 - \delta)\left(\frac{-\delta - 1 + \delta(1 - \delta\mu)}{(1 - \delta\mu)^2(1 - \mu)} \right) \\
&= -(1 - \delta)\left(\frac{1 + \delta^2\mu}{(1 - \delta\mu)^2(1 - \mu)} \right) \\
&< 0.
\end{aligned}
$$

Hence, there's a unique crossing point where the function $1 + \epsilon_\mu^{\mathcal{A}}$ is negative.

Finally, I compute relevant limits. Firs, we compute:

$$\lim_{\mu \to 0} 1 + \epsilon_\mu^{\mathcal{A}} = (1 - \delta)\lim_{\mu \to 0} \frac{\mu}{\ln\left(\frac{1 - \delta\mu}{1 - \mu} \right)} = \frac{1 - \delta}{\lim_{\mu \to 0} \frac{1 - \mu}{1 - \delta\mu}\frac{1 - \delta\mu}{1 - \mu}\left(\frac{-\delta}{1 - \delta\mu} - \frac{-1}{1 - \mu} \right)} = 1.$$

where the first equality are the surviving terms after taking limits, the second equality follows from L'Hospital's rule. For the limit $\mu \to 1$, by L'Hospital's rule:

$$\lim_{\mu \to 1} \frac{1}{1 - \mu}\left(\frac{1}{\ln\left(\frac{1 - \delta\mu}{1 - \mu} \right)} \right) = \lim_{\mu \to 1} \frac{-\frac{1}{1 - \mu^2}}{\frac{1 - \delta}{(1 - \delta\mu)(1 - \mu)}} = \lim_{\mu \to 1} -\frac{(1 - \delta\mu)}{(1 - \mu)(1 - \delta)}.$$

Adding terms the missing term:

$$
\begin{aligned}
\lim_{\mu \to 1} \left[1 + \epsilon_\mu^{\mathcal{A}} \right] &= \lim_{\mu \to 1} -\frac{(1 - \delta\mu)}{(1 - \mu)(1 - \delta)} - \frac{\mu}{1 - \mu} \\
&= \lim_{\mu \to 1} \frac{-1 + 2\delta\mu - \mu}{(1 - \mu)(1 - \delta)} \\
&= -2 \lim_{\mu \to 1} \frac{1}{(1 - \mu)} \\
&= -\infty.
\end{aligned}
$$

We finally compute the elasticity. From

$$
\frac{\mathcal{A}'(\mu)}{\mathcal{A}(\mu)} \mu < 0,
$$

we know that the elasticity $\epsilon_\mu^{\mathcal{A}}$ is decreasing. Hence, $1 + \epsilon_\mu^{\mathcal{A}}$ starts at zero and falls continuously until diverging at $\mu \to 1$.

B. Proofs of Section 3

B.1   Preliminary Observations

We begin with a set of identities that are convenient to proof the main results.

Average Price: Definitions and Identities.   Recall that the average price of a worker as it's expenditures relative to total consumption: $Q \equiv E^w / C^w$. The saver's expenditure is shown to be $E^s = (1 - \beta) B$. By the income expenditure identity:

$$E^w = 1 - (1 - \beta) B. \tag{22}$$

Total consumption by the worker, given its optimal expenditures, is given by:

$$C^w = \frac{E^w - S^w}{q} + S^w = \frac{E^w}{q} - \left(\frac{1}{q} - 1 \right) S^w.$$

where

$$S^w = \min \left\{ \max \left\{ 0, \tilde{B} - B \right\}, 1 - (1 - \beta) B \right\},$$

$X^w = (E^w - S^w) / q.$

Thus, we have that:

$$Q = \left(\frac{1}{q} - \left(\frac{1}{q} - 1 \right) \frac{S^w}{E^w} \right)^{-1} = \left(\frac{1}{q} \left(1 - \frac{S^w}{E^w} \right) + \frac{S^w}{E^w} \right)^{-1}.$$

Also, we have that:

$$Q = \frac{E^w - S^w}{C^w} + \frac{S}{C^w} = q \frac{X^w}{C^w} + \frac{S}{C^w}.$$

The last two proof the following result.

Lemma 3. *Q is average of the goods prices weighted by the worker's consumption shares and Q is the harmonic mean of the goods prices weighted by the expenditure share.*

Marginal Expenditure and Borrowing Prices.   For convenience, I define again the two prices that enter in marginal decisions. First, I define the *marginal expenditure*

price:

$$\tilde{q}_t^e \equiv q_t \mathbb{I}_{[B_t \geq B^\star(\tilde{B}_t)]} + \left(1 - \mathbb{I}_{[B_t \geq B^\star(\tilde{B}_t)]}\right).$$

Next, I define the *marginal borrowing price*:

$$\tilde{q}_{t+1}^b \equiv q_{t+1} \mathbb{I}_{[\tilde{B}_{t+1} \leq B_{t+1}]} + \left(1 - \mathbb{I}_{[\tilde{B}_{t+1} \leq B_{t+1}]}\right).$$

Analysis of the Marginal Expenditure Price. Next, we describe the behavior of $\tilde{q}_t^e$ in equilibrium. We have that $\tilde{q}_t^e = \tilde{q}^e\left(B_t, \tilde{B}_t\right)$ where:

$$\tilde{q}^e\left(B, \tilde{B}\right) \equiv q\left(\mu\left(B, \tilde{B}\right)\right) \mathbb{I}_{[B \geq B^\star(\tilde{B})]} + \left(1 - \mathbb{I}_{[B \geq B^\star(\tilde{B})]}\right).$$

In this expression, I am using:

$$\mu\left(B, \tilde{B}\right) = 1 - (1 - \beta) B - \min\left\{\max\left\{0, \tilde{B} - B\right\}, 1 - (1 - \beta) B\right\}.$$

Observe that

$$\mu\left(B, \tilde{B}\right) = \begin{cases} 0 & B < B^\star\left(\tilde{B}\right), \\ 1 + \beta B - \tilde{B} & B \in \left[B^\star\left(\tilde{B}\right), \tilde{B}\right], \\ 1 - (1 - \beta) B & B > \tilde{B}. \end{cases}$$

$\mu\left(B, \tilde{B}\right)$ is continuous, starts at zero and increases up to $B = \tilde{B}$, starting from that point, the function is decreasing. Since q_t is monotone in μ_t, the function $\tilde{q}^e\left(B, \tilde{B}\right)$ must follow the same pattern in the interior of the middle segment of teh function and in the last segment. Also notice that when $B < B^\star\left(\tilde{B}\right)$, $q^e = 1$. Next, we have that since μ is continuous, μ goes to 0 from above as $B \downarrow B^\star\left(\tilde{B}\right)$ so we can write,

$$\lim_{B \downarrow B^\star(\tilde{B})} \tilde{q}^e\left(B, \tilde{B}\right) = \lim_{\mu \downarrow 0} q\left(\mu\right) \cdot 1 = \delta^{-1}$$

where I used the fact that $\lim_{\mu \downarrow 0} \mathcal{A}(\mu) = \delta$. Hence, the function $\tilde{q}^e$ is discontinuous at $B = B^\star$ because $\lim_{B \uparrow B^\star(\tilde{B})} \tilde{q}^e = 1 \neq \delta^{-1}$, the function is also not monotonic. Then[27]

$$\tilde{q}^e\left(B, \tilde{B}\right) = \begin{cases} 1 & B < B^\star\left(\tilde{B}\right) \\ q & B \in \left[B^\star\left(\tilde{B}\right), \tilde{B}\right] \\ q & B > \tilde{B}. \end{cases}$$

Next, we are interested in the behavior of $\frac{Q}{q^e}$, for reasons that become clear in the main text. We have that for $B < B^\star\left(\tilde{B}\right)$, since $q = 1$, it must be that $\frac{Q}{q^e} = 1$. For $B \geq \tilde{B}$ also $q^e = Q = q$. Therefore, $\frac{Q}{q^e} = 1$. In the middle range of values, we have that:

$$\frac{Q}{q^e} = \frac{1}{q} \cdot \frac{q}{\left(1 - \frac{S^w}{E^w}\right) + q\frac{S^w}{E^w}} = \frac{1}{\left(1 - \frac{S^w}{E^w}\right) + q\frac{S^w}{E^w}}.$$

Thus, we have the following formula:

$$\frac{Q}{q^e} = \begin{cases} 1 & B < B^\star\left(\tilde{B}\right) \\ \frac{1}{1 - \frac{\min\{\max\{0, \tilde{B}-B\}, 1-(1-\beta)B\}}{1-(1-\beta)B} + q\left(\mu\left(B,\tilde{B}\right)\right)\frac{\min\{\max\{0, \tilde{B}-B\}, 1-(1-\beta)B\}}{1-(1-\beta)B}} & B \in \left[B^\star\left(\tilde{B}\right), \tilde{B}\right] \\ 1 & B > \tilde{B}. \end{cases}$$

Since at $B = \tilde{B}$ we have $\frac{S^w}{E^w} = 0$, the function is continuous at that point. However,

$$B \downarrow B^\star\left(\tilde{B}\right) \implies \frac{S^w}{E^w} \downarrow 1, \text{ and } q \downarrow \delta^{-1} > 1,$$

at that point. Thus, $\frac{Q}{q^e} = \delta$ at $B = B^\star\left(\tilde{B}\right)$ (and at its right limit) and $Q/q^e = 1$ at the left limit of this point. Namely, the function Q/q^e is discontinuous at $B^\star$.

Analysis of the Marginal Borrowing Price. Next, we investigate the behavior of the marginal borrowing price. Recall that this is the price of consumption at which the worker trades off future consumption when he borrows marginally. Let B_{t+1} be the

---

[27] In fact, the value of q^e at $B = B^\star$ will not be used since the function is discontinuous at that point and optimality conditions will be characterized via the right and left limit. However, I choose to define it according to the intuition presented lines above.

debt level the worker chooses today for next period and $\tilde{B}_{t+1}$ the next period's SBL. We have that $\tilde{q}^b_{t+1} = \tilde{q}^b\left(B_{t+1}, \tilde{B}_{t+1}\right)$ where:

$$\tilde{q}^b\left(B, \tilde{B}\right) \equiv q\left(\mu\left(B, \tilde{B}\right)\right) \mathbb{I}_{[B \geq \tilde{B}]} + \left(1 - \mathbb{I}_{[B \geq \tilde{B}]}\right).$$

Then,

$$\tilde{q}^b\left(B, \tilde{B}\right) = \begin{cases} 1 & B < B^\star\left(\tilde{B}\right), \\ 1 & B \in \left[B^\star\left(\tilde{B}\right), \tilde{B}\right], \\ q & B > \tilde{B}. \end{cases}$$

We have observed that for $B = B^\star\left(\tilde{B}\right)$, all consumption is spot and thus $q = 1$. However, at $B = \tilde{B}$, the function features a discontinuity since:

$$\lim_{B \downarrow \tilde{B}} \tilde{q}^b\left(B, \tilde{B}\right) = q\left(\mu\left(\tilde{B}, \tilde{B}\right)\right) = q\left(1 - (1 - \beta)\tilde{B}\right) > 1.$$

Next we investigate the behavior of $\frac{Q}{q^b}$.

For any $B < B^\star\left(\tilde{B}\right)$, both q^b and Q must equal 1, thus, $\frac{Q}{q^b} = 1$. Then, we have that for $B \in \left[B^\star\left(\tilde{B}\right), \tilde{B}\right]$, it must the case that $\frac{Q}{q^b} = Q$ because $q^b = 1$. Finally, when $B > \tilde{B}$, $\tilde{q}^b\left(B, \tilde{B}\right) = Q = q$. Thus, we have:

$$\frac{Q}{\tilde{q}^b} = \begin{cases} 1 & B < B^\star\left(\tilde{B}\right), \\ Q & B \in \left[B^\star\left(\tilde{B}\right), \tilde{B}\right], \\ 1 & B > \tilde{B}, \end{cases}$$

where

$$Q = \left(\frac{1}{q}\left(1 - \frac{S^w}{E^w}\right) + \frac{S^w}{E^w}\right)^{-1}$$

and $\lim_{B \downarrow B^\star\left(\tilde{B}\right)} Q = 1$.

Average Price Elasticity. Next, I derive the elasticity of the average price with respect to total debt, in equilibrium—i.e., after replacing $E^w = 1 - (1 - \beta) B$. This elasticity is critical for the Ramsey policy analysis.

We have that the average price is the harmonic mean:

$$Q = \frac{1}{\frac{1}{q} \left(1 - \frac{S^w}{E^w}\right) + \frac{S^w}{E^w}}.$$

where both the price q, E^w, and S^w are functions of B.

Thus, we have that:

$$\frac{\partial Q}{\partial B} = -Q \cdot \frac{\left(1 - \frac{S^w}{E^w}\right) \frac{\partial}{\partial B} \left[\frac{1}{q}\right] - \left(\frac{1}{q} - 1\right) \frac{\partial}{\partial B} \left[\frac{S^w}{E^w}\right]}{\left(\frac{1}{q} \left(1 - \frac{S^w}{E^w}\right) + \frac{S^w}{E^w}\right)}.$$

The numerator has two additional derivatives. The first one is:

$$\frac{\partial}{\partial B} \left[\frac{S^w}{E^w}\right] = \begin{cases} 0 & B < B^\star \left(\tilde{B}\right) \\ \left[\frac{\tilde{B} - B}{1 - (1 - \beta) B}\right] \left(\frac{(1 - \beta)}{1 - (1 - \beta) B} - \frac{1}{\tilde{B} - B}\right) & B \in [B^\star \left(\tilde{B}\right), \tilde{B}) \\ 0 & B \geq \tilde{B} \end{cases}$$

where the term in the intermediate region follows from:

$$\frac{\partial}{\partial B} \left[\frac{S^w}{E^w}\right] = \frac{S^w}{E^w} \left(\frac{1}{S^w} \frac{\partial}{\partial B} [S^w] - \frac{1}{E^w} \frac{\partial}{\partial B} [E^w]\right),$$

but then for $B \in [B^\star \left(\tilde{B}\right), \tilde{B})$

$$\frac{\partial}{\partial B} [S^w] = -1,$$

and

$$\frac{\partial}{\partial B} [E^w] = -(1 - \beta).$$

Hence:

$$\frac{\partial}{\partial B} \left[\frac{S^w}{E^w}\right] = \left[\frac{\tilde{B} - B}{1 - (1 - \beta) B}\right] \left(\frac{(1 - \beta)}{1 - (1 - \beta) B} - \frac{1}{\tilde{B} - B}\right) < 0.$$

We evaluate the limits of this function. Clearly, at

$$\frac{\partial}{\partial B} \left[\frac{S^w}{E^w} \right] \Bigg|_{B\uparrow\tilde{B}} = -\frac{1}{1 - (1-\beta)\tilde{B}} < -1,$$

and at

$$
\begin{aligned}
\frac{\partial}{\partial B} \left[\frac{S^w}{E^w} \right] \Bigg|_{B\downarrow B^\star} &= \frac{(1-\beta)\left(\tilde{B} - B^\star\left(\tilde{B}\right)\right)}{\left(1 - (1-\beta)B^\star\left(\tilde{B}\right)\right)^2} - \frac{1}{\left(1 - (1-\beta)B^\star\left(\tilde{B}\right)\right)} \\
&= \frac{(1-\beta)\left(\tilde{B} - B^\star\left(\tilde{B}\right)\right) - \left(1 - (1-\beta)B^\star\left(\tilde{B}\right)\right)}{\left(1 - (1-\beta)B^\star\left(\tilde{B}\right)\right)^2} \\
&= \frac{(1-\beta)\tilde{B} - 1}{\left(1 - (1-\beta)B^\star\left(\tilde{B}\right)\right)^2} \\
&= -1.
\end{aligned}
$$

The last steps follows from the definition of $B^\star$. Hence, this derivative is discontinuous. For the second derivative of interest, recall that:

$$\frac{\partial}{\partial B}[q] = \frac{\partial}{\partial \mu}[q] \cdot \frac{\partial \mu}{\partial B},$$

and

$$\frac{\partial \mu}{\partial B} = \frac{\partial}{\partial B}[E^w] - \frac{\partial}{\partial B}[S^w]$$

since $\mu = qX = E^w - S^w$. Then, we have that:

$$\frac{\partial}{\partial B}[q] = q_\mu \mu \frac{\frac{\partial}{\partial B}[E^w] - \frac{\partial}{\partial B}[S^w]}{E^w}.$$

We can express this as:

$$\frac{\partial}{\partial B}\left[q^{-1}\right] = \frac{1}{q}\epsilon_\mu^q \frac{\frac{\partial}{\partial B}[E^w] - \frac{\partial}{\partial B}[S^w]}{E^w - S^w}.$$

Hence, we obtain:

$$\frac{\partial Q}{\partial B}\frac{1}{Q} = -Q \cdot \left\{ \left(1 - \frac{S^w}{E^w}\right) \frac{1}{q}\epsilon_\mu^q \frac{\frac{\partial}{\partial B}\left[E^w\right] - \frac{\partial}{\partial B}\left[S^w\right]}{E^w - S^w} - \left(\frac{1}{q} - 1\right) \frac{S^w}{E^w} \left[\frac{\frac{\partial}{\partial B}\left[S^w\right]}{S^w} - \frac{\frac{\partial}{\partial B}\left[E^w\right]}{E^w}\right] \right\}.$$

B.2 Proof of Proposition 4 (Worker's Euler equation)

Recall also the relation between the $\tilde{B}_t$ and $B^\star_{t+1}$:

$$B^\star_{t+1} = R_{t+1}\left(\tilde{B}_t - 1\right).$$

The following lemma is used to reduce the set of cases we have to deal with.

Lemma 4. *Let $\tilde{B}_t$ be an increasing sequence and $\beta R_{t+1} \leq 1 \,\forall t$. Then, $B^\star_{t+1} \geq \tilde{B}_{t+1}$.*

Proof. Assume by contradiction that $B^\star_{t+1} \geq \tilde{B}_{t+1}$. Substituting the expression for $B^\star_{t+1}$, we have that

$$R_{t+1}\left(\tilde{B}_t - 1\right) = B^\star_{t+1} \geq \tilde{B}_{t+1} \geq \tilde{B}_t.$$

Hence,

$$(1 - 1/R_{t+1})\,\tilde{B}_t \geq 1.$$

If indeed $\beta \leq R^{-1}_{t+1}$, the condition above implies that:

$$(1 - \beta)\,\tilde{B}_t \geq 1.$$

However, this last inequality implies that:

$$\tilde{B}_t \geq \frac{1}{1 - \beta} = \bar{B}.$$

This is a contradiction.

$\square$

I now derive the worker's Euler equation, a necessary but not sufficient condition for optimality. Recall that we can write the worker's total expenditures as a function of B_{t+1}:

$$E^w_t = 1 - B_t + \frac{B_{t+1}}{R_{t+1}}.$$

Given her total expenditures, spot expenditures are:

$$S^w\left(B_t, \tilde{B}_t, B_{t+1}\right) = \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, 1 - B_t + \frac{B_{t+1}}{R_{t+1}}\right\}.$$

and her chained expenditures are:

$$q_t X_t^w = E_t^w - S^w \left(B_t, \tilde{B}_t, B_{t+1} \right).$$

Adding both types of expenditures dividing by the price, the worker's consumption is:

$$
\begin{aligned}
C_t &= \frac{1 - B_t + \frac{B_{t+1}}{R_{t+1}} - S^w \left(B_t, \tilde{B}_t \right)}{q_t} + S^w \left(B_t, \tilde{B}_t, B_{t+1} \right) \\
&= \frac{1 - B_t + \frac{B_{t+1}}{R_{t+1}}}{q_t} + \left(1 - \frac{1}{q_t} \right) \min \left\{ \max \left\{ \tilde{B}_t - B_t, 0 \right\}, 1 - B_t + \frac{B_{t+1}}{R_{t+1}} \right\}.
\end{aligned}
$$

Now consider a sequence $\{B_{t+1}\}_{t \geq 0}$. We obtain that the worker's problem can be written entirely in terms of the worker's debt level, without reference to his expenditures:

$$\sum_{t \geq 0} \beta^t \log \left(C_t \right) = \ldots$$

$$\sum_{t \geq 0} \beta^t \log \left(\frac{1 - B_t + \frac{B_{t+1}}{R_{t+1}}}{q_t} + \left(1 - \frac{1}{q_t} \right) \underbrace{\min \left\{ \max \left\{ \tilde{B}_t - B_t, 0 \right\}, 1 - B_t + \frac{B_{t+1}}{R_{t+1}} \right\}}_{S^w \left(B_t, \tilde{B}_t, B_{t+1} \right)} \right).$$

There are two kinks in the term $S^w \left(B_t, \tilde{B}_t, B_{t+1} \right)$. These kinks occur at the threshold points given in Lemma 1, the points $\left\{ \tilde{B}_t, B_{t+1}^\star \right\}$. Since the control variable in this problem is B_{t+1}, we have to consider the kinks. The first kink, $\tilde{B}_{t+1}$, corresponds to the value of B_{t+1} to the left of that value, consumption in time $t + 1$ features zero spot consumption. The second kink, $B_{t+1}^\star$, corresponds to the level of debt B_{t+1} from which to its right, there is some chained consumption in time t.

Consider two consecutive periods in the worker's optimal sequence:

$$\log \left(C_t \right) + \beta \log \left(C_{t+1} \right)$$

Define the function

$$\Upsilon_t \left(B_{t+1}; B_t^o, B_{t+2}^o, \tilde{B}_t, \tilde{B}_{t+1}, R_{t+1}, q_t \right) \equiv$$

$$\log \left(\frac{1 - B_t^o + \frac{B_{t+1}}{R_{t+1}}}{q_t} + \left(1 - \frac{1}{q_t}\right) \min \left\{ \max \left\{ \tilde{B}_t - B_t^o, 0 \right\}, 1 - B_t^o + \frac{B_{t+1}}{R_{t+1}} \right\} \right) +$$

$$\beta \log \left(\frac{1 - B_{t+1} + \frac{B_{t+2}^o}{R_{t+2}}}{q_{t+1}} + \left(1 - \frac{1}{q_{t+1}}\right) \min \left\{ \max \left\{ \tilde{B}_{t+1} - B_{t+1}, 0 \right\}, 1 - B_{t+1} + \frac{B_{t+2}^o}{R_{t+2}} \right\} \right).$$

The function Υ_t represents the value of utility at t and $t + 1$, considering the optimal choices $\left\{ B_t^o, B_{t+2}^o \right\}$, for an arbitrary level of debt B_{t+1}. An optimal solution must satisfy:

$$\log \left(C_t \right) + \beta \log \left(C_{t+1} \right) = \max_{B_{t+1}} \Upsilon_t \left(B_{t+1}; B_t^o, B_{t+2}^o, \tilde{B}_t, \tilde{B}_{t+1}, R_{t+1}, q_t \right).$$

Thus, we use a perturbation argument, with respect to B_{t+1}, to derive a generalized Euler equation. In all points $B_{t+1} \in \left(0, \overline{B}\right)$ other than the threshold points, the objective is continuous, locally concave and differentiable in B_{t+1}. The kinks are in fact points of no differentiability—because of a discontinuity of the derivative of Υ_t with respect to B_{t+1}.

Let's consider the differentiability points first and then deal with the kinks. The objective of the first terms is increasing in B_{t+1}. The objective of the second term is decreasing. Thus, in the intervals determined by the kinks, marginal benefits and costs of increasing B_{t+1} cross at most at a single point. At the kinks, the derivatives feature discontinuities, hence, multiple critical points may arise. I present the analysis of the critical points.

I break the analysis into each of the following cases.

I. Let $B_t^o \geq \tilde{B}_t$.

I.a $B_{t+1} < \tilde{B}_{t+1}$, the derivative is:

$$\Upsilon_t' \left(B_{t+1} \right) = \frac{1}{C_t} \frac{1}{q_t R_{t+1}} - \beta \frac{1}{C_{t+1}},$$

regardless of whether $B_{t+1} > B_{t+1}^\star$.

I.b $B_{t+1} > \tilde{B}_{t+1}$, the derivative of Υ_t with respect to B_{t+1} is:

$$\Upsilon'_t(B_{t+1}) = \frac{1}{C_t}\frac{1}{q_t R_{t+1}} - \beta\frac{1}{C_{t+1}}\frac{1}{q_{t+1}}$$

I.a-I.b. Combining both case, observe that at $\tilde{B}_{t+1}$ the following strict inequality holds

$$\lim_{B'\uparrow\tilde{B}_{t+1}} \Upsilon'_t(B') < \lim_{B'\downarrow\tilde{B}_{t+1}} \Upsilon'_t(B') \tag{23}$$

From the right of $\tilde{B}_{t+1}$, the value of forgone consumption at $t+1$ given an increase in B_{t+1} at t, is lower due to the higher price of consumption at $t + 1$. As a result, if limit form the right of $\Upsilon'_t \leq 0$ there is no critical point to the right of $\tilde{B}_{t+1}$ and if $\Upsilon'_t \geq 0$ from the left then there is no critical point to the left of $\tilde{B}_{t+1}$.

In summary, when $B^o_t \geq \tilde{B}_t$, and there is only consumption of chained goods:

- If $B' \in \left(0, \tilde{B}_{t+1}\right)$ is a local maximum, then $\Upsilon'_t = 0$. If furthermore

$$\lim_{B'\downarrow\tilde{B}_{t+1}} \Upsilon'_t(B') \leq 0,$$

  then only one possible value of B_{t+1} satisfies the Euler equation.

- If $B' \in \left(\tilde{B}_{t+1}, \overline{B}\right)$ is a local maximum, then $\Upsilon'_t = 0$. If furthermore

$$\lim_{B'\uparrow\tilde{B}_{t+1}} \Upsilon'_t(B') \geq 0,$$

  then only one possible value of B_{t+1} satisfies the Euler equation.

- Since the objective is concave on both intervals, there's only one possible solution to B'.

- $B = \tilde{B}_{t+1}$ is not a solution since this requires $\lim_{B'\uparrow\tilde{B}_{t+1}} \Upsilon'_t(B') \geq 0 \geq \lim_{B'\downarrow\tilde{B}_{t+1}} \Upsilon'_t(B')$, which contradicts (23).

Hence, we have shown the following lemma.

Lemma 5. *When $B_t \geq \tilde{B}_t$, B_{t+1} satisfies the Euler equation with equality:*

$$\Upsilon_t'(B_{t+1}) = 0.$$

and $B_{t+t} = \tilde{B}_{t+1}$ is not a solution.

II. Let $B_t^o < \tilde{B}_t$ in a solution to the worker's problem. We know by the Lemma above that $B_{t+1}^\star < \tilde{B}_{t+1}$. Hence, we ave the following cases:

II.a $B_{t+1} < B_{t+1}^\star < \tilde{B}_{t+1}$, there is only spot consumption at t and some spot consumption at $t + 1$, hence the derivative of the objective is:

$$\Upsilon'(B_{t+1}) = \frac{1}{C_t} \cdot \frac{1}{R_{t+1}} - \beta \frac{1}{C_{t+1}}.$$

II.b For $B_{t+1} \in \left(B_{t+1}^\star, \tilde{B}_{t+1}\right)$ there is some chained consumption at t and some spot consumption at $t + 1$, hence the derivative of the objective is:

$$\Upsilon_t'(B_{t+1}) = \frac{1}{q_t} \frac{1}{C_t} \cdot \frac{1}{R_{t+1}} - \beta \frac{1}{C_{t+1}}.$$

II.c. For $B_{t+1}^\star < \tilde{B}_{t+1} < B_{t+1}$ there is some chained consumption at t and no spot consumption at $t + 1$, hence the derivative of the objective is:

$$\Upsilon_t'(B_{t+1}) = \frac{1}{C_t} \cdot \frac{1}{q_t} \frac{1}{R_{t+1}} - \beta \frac{1}{C_{t+1}} \frac{1}{q_{t+1}}.$$

II.a.-II.c At then not differentiable points the following strict inequalities hold

$$\lim_{B' \uparrow B_{t+1}^\star} \Upsilon_t'(B') > \lim_{B' \downarrow B_{t+1}^\star} \Upsilon_t'(B')$$

and

$$\lim_{B' \uparrow \tilde{B}_{t+1}} \Upsilon_t'(B') < \lim_{B' \downarrow \tilde{B}_{t+1}} \Upsilon_t'(B').$$

where the inequalities follow the same arguments as in case I.

- If $B \in \left(0, B^\star \left(R_{t+1}, \tilde{B}_t\right)\right)$ is a solution then $\Upsilon'_t (B) = 0$. If furthermore

$$\lim_{B' \uparrow B^\star_{t+1}} \Upsilon'_t (B') \leq 0,$$

  then only possible value of B_{t+1} satisfies the Euler equation.

- If $B = B^\star \left(R_{t+1}, \tilde{B}_t\right)$ is a solution then

$$\lim_{B' \uparrow B^\star_{t+1}} \Upsilon'_t (B') \geq 0 \geq 0 \lim_{B' \downarrow B^\star_{t+1}} \Upsilon'_t (B')$$

  where at most one inequality is strict.

- If $B \in \left(B^\star \left(R_{t+1}, \tilde{B}_t\right), \tilde{B}_{t+1}\right)$ is a solution then $\Upsilon'_t (B) = 0$. If furthermore

$$\lim_{B' \downarrow B^\star_{t+1}} \Upsilon'_t (B') \leq 0,$$

  then only possible value of B_{t+1} satisfies the Euler equation.

- $\tilde{B}_{t+1}$ is not a solution as it yields a contradiction.

- If $B \in \left(\tilde{B}_{t+1}, \overline{B}\right)$ is a solution then $\Upsilon'_t (B) = 0$. If furthermore $\lim_{B' \uparrow \tilde{B}_{t+1}} \Upsilon'_t (B')$, then only possible value of B_{t+1} satisfies the Euler equation.

- Again, by concavity, we have a unique path in each case.

We thus have shown the following lemma:

Lemma 6. *When $B_t < \tilde{B}_t$, B_{t+1} either satisfies the Euler equation with equality:*

$$\Upsilon'_t (B_{t+1}) = 0.$$

or $B' = B^\star_{t+1}$ if:

$$\lim_{B' \uparrow B^\star_{t+1}} \Upsilon'_t (B') \geq 0 \geq \lim_{B' \downarrow B^\star_{t+1}} \Upsilon'_t (B').$$

In the latter case, there's only spot consumption at t.

Necessity. Using the definition of Q_t, we have:

$$\frac{1}{C_t} \cdot \frac{\Pi_{t+1}(B_{t+1})}{R_{t+1}} - \beta \frac{1}{C_{t+1}} = \frac{Q_t}{E_t} \cdot \frac{\Pi_{t+1}(B_{t+1})}{R_{t+1}} - \beta \frac{Q_{t+1}}{E_{t+1}}.$$

The cases above, are captured by the term $\Pi_{t+1}(B_{t+1})$. Hence, the equation above yields the sign of the derivative of the change in B_{t+1}. Moreover,

$$\Upsilon_t'(B') \geq 0 \rightarrow \frac{Q_t}{E_t} \cdot \frac{1}{R_{t+1}} - \beta \frac{Q_t}{C_{t+1}} \geq 0 \rightarrow \frac{E_{t+1}}{E_t} \cdot \frac{Q_t}{Q_{t+1}} \geq \beta \frac{R_{t+1}}{\Pi_{t+1}(B_{t+1})}.$$

and vice versa. Thus:

$$\lim_{B' \uparrow \tilde{B}_{t+1}} \Upsilon_t'(B') \geq 0 \rightarrow \frac{E_{t+1}}{E_t} \geq \beta R_{t+1}$$

and

$$\lim_{B' \downarrow B_{t+1}^\star} \Upsilon_t'(B') \leq 0 \rightarrow q_t \beta R_{t+1} \geq \frac{E_{t+1}}{E_t}$$

Collecting all the cases above and using the definition of Π_t, we arrive at a more general version of the proposition that I show in the text.

Proposition 10. (Workers's First-Order Condition): *Fix a sequence $\left\{ \tilde{B}_t, R_{t+1}, q_t \right\}_{t \geq 0}$ such that $\tilde{B}_t$ is an increasing and $\beta R_{t+1} \leq 1$. Then, any solution $\{B_{t+1}\}_{t \geq 0}$ to the worker's problem satisfies the following generalized Euler equation:*

$$\frac{E_{t+1}^w}{E_t^w} \frac{Q_t}{Q_{t+1}} = \beta \frac{R_{t+1}}{\Pi_{t+1}(B_{t+1})} \quad if \quad B_{t+1} \neq B_{t+1}^\star \tag{24}$$

and

$$q_t \beta R_{t+1} \geq \frac{E_{t+1}}{E_t} \geq \beta R_{t+1} \quad if \quad B_{t+1} = B_{t+1}^\star.$$

If $B_{t+1} = B_{t+1}^\star$, then there is only spot consumption at t and $Q_t = 1$.

B.3   Proof of Proposition 5 (Worker's Solution)

In a stationary version of the worker's problem, $R_t = \beta^{-1}$, the SBL is constant, $\tilde{B}_t = B_{ss}$, and the chained-goods price is constant, $q_t = q$. In the rest of the proof, we use $\tilde{B}_{ss} > 1$. For convenience, is suppress the arguments of $B^\star \left(1/\beta, \tilde{B}_{ss} \right)$ to define its steady state value, $B^\star$. I set to prove the following:

I. If $B_0 \in [0, B^\star]$, then $B_t = B_0, \forall t$.

II. There exists a threshold $B^h > \tilde{B}_{ss}$ such that:

II.a If $B_0 < B^h$, then $B_t \to B^\star$ in finite time.

II.b If $B_0 > B^h$, then $B_t = B_0 \,\forall t$.

The proof presented in this Appendix, also characterizes the convergence times $B_t \to B^\star \left(1/\beta, \tilde{B}_{ss} \right)$ and the threshold B^h. The full characterization is discussed in the following section of this appendix, together with a graph.

   The strategy is as follows: First, I present two value functions, $\bar{V}(B)$ and $\underline{V}(B)$, that correspond to upper and lower bounds of the worker's value function. To prove Part I, I show that for $B_0 \leq B^\star$ a constant debt policy is optimal and delivers the same value us the upper bound $\bar{V}(B)$. To prove Part II.a, I show that for any $B_0 \in (B^\star \left(\tilde{B} \right), \tilde{B}]$, the optimal solution implies deleveraging in finite time where debt converges to $B_t \to B^\star$. This implies that $V(B) > \underline{V}(B)$ in $B_0 \in (B^\star \left(\tilde{B} \right), \tilde{B}]$. Finally, to prove Part II.a and II.b, I prove the existence of a level $B^h > \tilde{B}$ such that for debt levels above B^h there worker leaves his debt constant and $V(B) = \underline{V}(B)$. I finally show that if $V(B) > \underline{V}(B)$, then, $V(\cdot) > \underline{V}(\cdot)$ holds for any debt level below that value. This guarantees the existence of a unique threshold B^h.

   As a preliminary calculation, I solve for two value functions that correspond to upper and lower bounds of the worker's value function.

Value Function Bounds.   I consider two auxiliary problems. First, the unconstrained worker problem that produces an upper bound:

Problem 7.

$$\bar{V}(B) = \max_{B' \leq \bar{B}} \ln(E^w) + \beta \bar{V}(B')$$

subject to:

$$B + E^w = 1 + \frac{B'}{R}.$$

Second, a worker problem where the worker is induced to always consume chained goods:

Problem 8.

$$\underline{V}(B) = \max_{B' \leq \bar{B}} \ln(E^w/q) + \beta \underline{V}(B')$$

subject to:

$$B + E^w = 1 + \frac{B'}{R}.$$

The respective solutions to these problems is given in the Following Lemma.

Lemma 7. *The solutions to $\bar{V}(B)$ and $\underline{V}(B)$ are:*

$$\bar{V}(B) = \frac{\ln(1 - (1 - \beta)B)}{1 - \beta}$$

and

$$\underline{V}(B) = \frac{\ln(1 - (1 - \beta)B) - \ln(q)}{1 - \beta}.$$

In both cases, optimal expenditures are:

$$E^w = 1 - (1 - \beta)B.$$

Proof. I first solve for $\underline{V}$. I guess and verify that the solution is:

$$\underline{V}(B) = \frac{\ln(1 - (1 - \beta)B) - \ln(q)}{1 - \beta}.$$

Using the guess into the value function:

$$\underline{V}(B) = \max_{B' \leq \bar{B}} \ln\left(1 - B + \frac{B'}{R}\right) - \ln(q) + \frac{\beta}{1 - \beta}\ln(1 - (1 - \beta)B') + \frac{\beta}{1 - \beta}\ln(q).$$

Taking first-order conditions with respect to B' yields:

$$\frac{1}{1 - B + B'/R}\frac{1}{R} = \frac{\beta}{1 - (1 - \beta)B'}.$$

Then, using that $\beta = R^{-1}$, we obtain:

$$\frac{1}{1 - B + B'\beta} = \frac{1}{1 - B' + B'\beta}$$

Thus, the solution is $B = B'$.

As a result, expenditures are:

$$C = (1 - (1 - \beta) B) q.$$

$$E^w = 1 + \left(\frac{B}{R} - B\right) = 1 - (1 - \beta) B.$$

Substituting back expenditures into the Bellman equation, we obtain:

$$\begin{aligned}\underline{V}(B) &= \ln(1 - (1 - \beta) B) - \ln(q) + \frac{\beta}{1 - \beta}\left(\ln(1 - (1 - \beta) B) - \ln(q)\right) \\ &= \frac{\ln(1 - (1 - \beta) B)}{1 - \beta} - \frac{\ln(q)}{1 - \beta}\end{aligned}$$

which verifies the conjecture. Specializing to $q = 1$, we also obtain the solution for $\bar{V}(B)$.

$\square$

Note that $V(B) \in \left[\underline{V}(B), \bar{V}(B)\right]$, since the value functions correspond to more constrained and more relaxed problems than the original worker's problem.

Proof of Part I. I now proof Part I in the statement of the proposition. Part I is a special case of the following Lemma.

Lemma 8. *For $B \leq B^{\star}(\tilde{B})$, $V(B) = \bar{V}(B)$ and $E^w = 1 - (1 - \beta) B$.*

Proof. The proof is immediate. We know that $V(B) \leq \bar{V}(B)$. I guess and verify that setting

$$E^w = 1 - (1 - \beta) B$$

in the original problem, yields the same value.

Note that:

$$
\begin{aligned}
E^w &\leq 1 - (1 - \beta) B^\star \\
&= 1 + \beta B^\star - B^\star \\
&= 1 + \beta \beta^{-1} \left(\tilde{B} - 1 \right) - B^\star(\tilde{B}) \\
&= \tilde{B} - B^\star(\tilde{B}) \\
&\leq \tilde{B} - B.
\end{aligned}
$$

The first inequality follows because $B < B^\star$, the second equality uses the definition of $B^\star$. The last equality proofs that consuming only spot is feasible. Moreover,

$$
B' = \beta^{-1} \left(B + E^w - 1 \right) = B.
$$

Since consumption is:

$$
C = 1 - (1 - \beta) B,
$$

for all periods, $V(B)$, attains the upper bound,

$$
V(B) = \frac{\ln (1 - (1 - \beta) B)}{1 - \beta} = \bar{V}(B).
$$

$\square$

Proof of Part II - Preliminary Lemmas. I now begin the proof of Parts II.a and II.b in the proposition. A key result is to find the set of values B for which setting $B' = B^\star$ is optimal. First, I present an intermediate Lemma:

Lemma 9. *If $B' = B^\star$, then $C = \tilde{B} - B$.*

Proof. Whenever $B' = B^\star$ we have, by definition of $B^\star$, that $Q = 1$. Thus:

$$
C = 1 + \frac{B^\star}{R} - B = 1 + \frac{\left(\tilde{B} - 1 \right)}{\beta R} - B = \tilde{B} - B, \tag{25}
$$

where I used that: $B^\star = 1/\beta \left(\tilde{B} - 1 \right).$

$\square$

Next, I find the values of B for which setting $B' = B^\star$ is satisfies the worker's Euler equation, (13). Define the set:

$$\mathcal{B}_0 \equiv \left[B^\star, B^\star + \left(1 - \frac{1}{q} \right) \left(\tilde{B} - B^\star \right) \right]. \tag{26}$$

This set defines the set of values of B that converge to $B^\star$ after one period.

Proposition 11. $B' = B^\star$ *if and only if* $B \in \mathcal{B}_0$.

Proof. To proof the result, I evaluate find the set of values B' such that the sub-differential equation for the Euler equation holds:

$$\frac{C \left(B^\star \right)}{C \left(B' \right)} \in [1, q],$$

where I used that marginal prices are both 1 since $B' \leq B^\star < \tilde{B}$ and $\beta R = 1$. Hence, by the Euler equation is:

$$\frac{\tilde{B} - B^\star}{\tilde{B} - B} \geq 1 \iff B \geq B^\star,$$

so $B \geq B^\star$ and, thus, $B^\star$ is the lower bound of the interval. For the upper bound I use that:

$$\frac{\tilde{B} - B^\star}{\tilde{B} - B} \leq q \iff B \leq \tilde{B} - \frac{1}{q} \left(\tilde{B} - B^\star \right) < \tilde{B}.$$

Hence, for any $B \in \mathcal{B}_0$, the inequality condition that replaces the Euler equation is satisfied.

□

I show another auxiliary result regarding a sequence of debt that delevers to reach the set $B \in \mathcal{B}_0$ while satisfying the Euler equation:

Proposition 12. *Let* $\gamma \equiv \beta/q$. *Define the function:*

$$\Gamma(t) \equiv \frac{1 - \gamma^t}{1 - \gamma} - q^{-t} \frac{1 - \beta^t}{1 - \beta}, \quad t \geq 0.$$

Let T be the unique positive integer that satisfies:

$$\Gamma(T) < 1 < \Gamma(T+1).$$

Any decreasing sequence B_{-t} for $t \in \{0, T\}$, that satisfies the worker's Euler equation, (13), with a terminal condition $B_0 \in \mathcal{B}_0$, satisfies that $B_{-t} \in \mathcal{B}_{-t}$ where

$$\mathcal{B}_{-t} \equiv (B_{-t}^{\ell}, B_{-t-1}^{\ell}] \cap \left[0, \tilde{B}_{ss}\right],$$

and where,

$$B_{-t}^{\ell} \equiv B^{\star} + \left(\tilde{B} - B^{\star}\right)\Gamma(t).$$

Moreover, $\cup_{t=0}^{T}\mathcal{B}_{-t} = (B^{\star}, \tilde{B}_{ss}]$.

The proposition presents a set of sequences that solve the Euler equation backwards in time.

Proof. For any $B \in (B^{\star}, \tilde{B}]$, the ratio of marginal prices is q. Consider a decreasing sequence of debt levels that converges to some $B_0 \in \mathcal{B}_0$. Then, for any $B_{-t} \in (B^{\star}, \tilde{B}_{ss}]$, we obtain a difference equation induced the worker's Euler equation:

$$C_{-t} = \frac{1}{q}C_{-t+1} \text{ if } B_{-t+1} \in (B^{\star}, \tilde{B}_{ss}], \tag{27}$$

for $t \in [1, T]$, and satisfies the terminal condition $C_0 = \tilde{B} - B_0$. Shifting forward the Euler equation (27) I obtain:

$$C_0/C_{-t} = q^t. \tag{28}$$

Next, I transform the Euler equation into a difference equation in debt. Using the budget constraint:

$$C_{-t} = \underbrace{\tilde{B} - B_{-t}}_{S} + \underbrace{\frac{1}{q}\left(1 + \beta B_{-t+1} - \tilde{B}\right)}_{X} \text{ if } B_{-t+1} \in (B^{\star}, \tilde{B}_{ss}]. \tag{29}$$

In particular,

$$C_0 = \tilde{B} - B_0, \tag{30}$$

by using the definition of $B^\star$. Hence, combining (29) with (30), and (28), we arrive at:

$$\tilde{B} - B_{-t} + q^{-1}\left(1 + \beta B_{-t+1} - \tilde{B}\right) = q^{-t}\left(\tilde{B} - B_0\right), \quad \forall t \in [1, T],$$

where T is the highest value of t such that $B_{-T} < \tilde{B}$.

Re-arranging, we obtain

$$B_{-t} = \tilde{B} - q^{-t}\left(\tilde{B} - B_0\right) + q^{-1}\left(1 + \beta B_{-t+1} - \tilde{B}\right), \quad \forall t \in [1, T].$$

We can simplify this expression by noticing that:

$$\begin{aligned}
1 + \beta B_{-t+1} - \tilde{B} &= 1 + \beta B_{-t+1} + \beta B^\star - \beta B^\star - \tilde{B} \\
&= \beta\left(B_{-t+1} - B^\star\right) + 1 + \beta B^\star - \tilde{B} \\
&= \beta\left(B_{-t+1} - B^\star\right).
\end{aligned}$$

Thus, we produce a difference equation in debt:

$$B_{-t} = \tilde{B} - q^{-t}\left(\tilde{B} - B_0\right) + \frac{\beta}{q}\left(B_{-t+1} - B^\star\right), \quad \forall t \geq 1. \tag{31}$$

This difference equation must have a terminal condition in some $B_0 \in \mathcal{B}_0$.

Let $\{B^\ell_{-t}\}$ and $\{B^u_{-t}\}$ denote the boundaries of the set $\mathcal{B}_{-t}$:

$$B^u_0 \equiv B^\star + \left(1 - \frac{1}{q}\right)\left(\tilde{B} - B^\star\right)$$

and

$$B^\ell_0 \equiv B^\star.$$

Lemma 10. *Let $B_0 = B^\ell_0$. The solution to (31)*

$$B^\ell_{-t} = \tilde{B} - q^{-t}\left(\tilde{B} - B_0\right) + \frac{\beta}{q}\left(B^\ell_{-t+1} - B^\star\right), \quad \forall t \geq 1$$

is:

$$B^\ell_{-t} = B^\star + \left(\tilde{B} - B^\star\right)\sum_{\tau=0}^{t-1}\gamma^\tau\left(1 - q^{-t+\tau}\right).$$

where $\gamma \equiv \beta / q$.

Proof. Subtract $B^\star$ from both sides of the difference equation (31):

$$B^\ell_{-t} - B^\star = \tilde{B} - B^\star - q^{-t}\left(\tilde{B} - B_0\right) + q^{-1}\beta\left(B^\ell_{-t+1} - B^\star\right), \quad \forall t \geq 1.$$

Define $Z^\ell_{-t} \equiv B_{-t} - B^\star$. With this change of variables, we obtain the difference equation:

$$Z_{-t} = \tilde{B} - B^\star - q^{-t}\left(\tilde{B} - B^\star\right) + q^{-1}\beta Z^\ell_{-t+1}, \quad \forall t \geq 1,$$

with terminal condition $Z^\ell_0 = 0$. Therefore, since $q^{-1}\beta < 1$, the series converges to:

$$Z_{-\infty} = \frac{\tilde{B} - B^\star}{1 - q^{-1}\beta} > \tilde{B} - B^\star \rightarrow B_{-\infty} = B^\star + \frac{\tilde{B} - B^\star}{1 - q^{-1}\beta} > \tilde{B},$$

implying that the equation is valid only up to some T.

I now solve the difference equation, rolling the difference forward in time up to $t = 0$:

$$
\begin{aligned}
Z_{-t} &= \tilde{B} - B^\star - q^{-t}\left(\tilde{B} - B^\star\right) + q^{-1}\beta Z_{-t+1} \\
&= \left(\tilde{B} - B^\star - q^{-t}\left(\tilde{B} - B^\star\right)\right) + q^{-1}\beta\left(\left(\tilde{B} - B^\star - q^{-t+1}\left(\tilde{B} - B^\star\right)\right) + q^{-1}\beta Z_{-t+2}\right) \\
&= \sum_{\tau=0}^{t-1}\left(q^{-1}\beta\right)^\tau\left(\left(\tilde{B} - B^\star\right) - q^{-t+\tau}\left(\tilde{B} - B^\star - Z_0\right)\right) + \left(q^{-1}\beta\right)^t Z_0.
\end{aligned}
$$

For convenience, use $\gamma \equiv \beta / q$. We produce:

$$
\begin{aligned}
Z_{-t} &= \sum_{\tau=0}^{t-1}\gamma^\tau\left(\left(\tilde{B} - B^\star\right) - q^{-t+\tau}\left(\tilde{B} - B^\star - Z_0\right)\right) + \gamma^t Z_0, \\
&= \left(\tilde{B} - B^\star\right)\sum_{\tau=0}^{t-1}\gamma^\tau\left(1 - q^{-t+\tau}\right) + Z_0\left(\sum_{\tau=0}^{t-1}\gamma^\tau q^{-t+\tau} + \gamma^t\right).
\end{aligned}
$$

Since $Z_0 = 0$ we obtain:

$$Z_{-t} = \left(\tilde{B} - B^\star\right)\sum_{\tau=0}^{t-1}\gamma^\tau\left(1 - q^{-t+\tau}\right).$$

This condition implies:

$$B^{\ell}_{-t} = B^{\star} + \left(\tilde{B} - B^{\star} \right) \sum_{\tau=0}^{t-1} \gamma^{\tau} \left(1 - q^{-t+\tau} \right)$$

as long as $B^{\ell}_{-t} < \tilde{B}$.

□

Notice that $B^{\ell}_{-1} = B^{\star} + \left(1 - \frac{1}{q} \right) \left(\tilde{B} - B^{\star} \right) = B^{u}_{0}$. Hence, the following corollary is immediate:

Corollary 4. *Let* $B_0 = B^{u}_0$. *The solution to (31)*

$$B^{u}_{-t} = \tilde{B} - q^{-t} \left(\tilde{B} - B_0 \right) + \frac{\beta}{q} \left(B_{-t+1} - B^{\star} \right), \quad \forall t \geq 1$$

is:

$$B^{u}_{-t} = B^{\star} + \left(\tilde{B} - B^{\star} \right) \sum_{\tau=0}^{t} \gamma^{\tau} \left(1 - q^{-t+\tau} \right) = B^{\ell}_{-t-1}.$$

where $\gamma \equiv \beta/q$.

This sequences $\{ B^{\ell}_{-t} \}$ and $\{ B^{u}_{-t} \}$ characterizes the boundary of the sets $\mathcal{B}_{-t}$ such that $B_{-t} \in \mathcal{B}_{-t}$, for $t \geq 0$. To see this, note that because $B^{u}_{t} = B^{\ell}_{-t-1}$, the sequence of intervals:

$$\mathcal{B}_{-t} \equiv (B^{\ell}_{-t}, B^{\ell}_{-t-1}] \cap \left[0, \tilde{B}_{ss} \right], \quad t \in \{0, T\},$$

where,

$$B^{\ell}_{-t} \equiv B^{\star} + \left(\tilde{B} - B^{\star} \right) \Gamma(t).$$

form $\cup_{t=0}^{T} \mathcal{B}_{-t} = (B^{\star}, \tilde{B}_{ss}]$. Finally, observe that any B_{-t} given by (31) is monotonically increasing in B_{-t-1}. Hence, any $B_{-t} \in \mathcal{B}_{-t}$.

The summation term:

$$\sum_{\tau=0}^{t-1} \gamma^{\tau} \left(1 - q^{-t+\tau} \right) = \frac{1 - \gamma^t}{1 - \gamma} - q^{-t} \frac{1 - \beta^t}{1 - \beta}.$$

Define:

$$\Gamma(t) \equiv \frac{1 - \gamma^t}{1 - \gamma} - q^{-t} \frac{1 - \beta^t}{1 - \beta}.$$

I use this function to find T. The term, $B^\ell_{-t} \in (B^\star, \tilde{B}]$ if and only if $\Gamma(t) < 1$. Hence, the last t for which $(B^\ell_{-t}, B^\ell_{-t-1}] \cap \left[0, \tilde{B}_{ss}\right] \neq \emptyset$, satisfies:

$$\Gamma(t) \equiv \frac{1 - \gamma^t}{1 - \gamma} - q^{-t}\frac{1 - \beta^t}{1 - \beta} < 1 < \frac{1 - \gamma^{t+1}}{1 - \gamma} - q^{-t+1}\frac{1 - \beta^{t+1}}{1 - \beta} = \Gamma(t+1). \tag{32}$$

Indeed, there is a unique integer T that satisfies the condition. To see this, note that for $\Gamma(1) = 1 - q^{-1} < 1$, but $\lim_{t\to\infty} \Gamma(t) = 1/(1 - \gamma) > 1$. In turn, the function is increasing in t. Since there is a unique positive integer T that satisfies the condition above, there is a maximum interval $\mathcal{B}_{-T} = (B_{-T}, \tilde{B}]$ that satisfies the Euler equation in $(B^\star, \tilde{B}]$. Naturally, $\mathcal{B}_{-t}$, for $t \in \{0, 1, ..., T\}$ defines a partition of $\left[B^\star, \tilde{B}\right]$.

$\square$

The implication of the Lemma above is that any deleveraging sequence that converges to $B^\star$, that at some point crosses $(B^\star, \tilde{B}]$, satisfies the property that each point of the sequence B_{-t} for $t \in \{0, 1, ..., T\}$, falls in in the corresponding interval $\mathcal{B}_{-t}$. In turn, there are at most a finite number T of such intervals. For any t, B_{-t} reaches B_0 in t periods and $B^\star$ in $t+1$ periods.

Next, I show an analogue Proposition for debt levels above $\tilde{B}$.

Proposition 13. *Let T, and B^ℓ_{-T} be given by Proposition 12. Any decreasing sequence B_{-t} for $t \in \{0, \infty\}$, that satisfies the worker's Euler equation, (13), with terminal condition $B_0 \in \mathcal{B}_0$, satisfies that*

$$B_{-T-\tau} \in \left[B^\ell_{-T-\tau}, B^u_{-T-\tau}\right], \quad \forall \tau \geq 1. \tag{33}$$

where

$$B^\ell_{-T-\tau} \equiv \frac{\left(1 - q^{-T}\left(\tilde{B} - B^\star\right)\right)}{1 - \beta} + \beta^\tau \left(B^\ell_{-T} - \frac{\left(1 - q^{-T}\left(\tilde{B} - B^\star\right)\right)}{1 - \beta}\right),$$

and

$$B^u_{-T-\tau} \equiv \frac{\left(1 - q^{-T}\left(\tilde{B} - B^\star\right)\right)}{1 - \beta} + \beta^\tau \left(B^\ell_{-T-1} - \frac{\left(1 - q^{-T}\left(\tilde{B} - B^\star\right)\right)}{1 - \beta}\right),$$

Proof. Consider the sets:

$$\mathcal{B}_{-T} = \left[B^{\ell}_{-T}, \tilde{B} \right] \quad \text{and} \quad \tilde{\mathcal{B}}_{-T} = (\tilde{B}, B^{\ell}_{-T-1}],$$

where

$$B^{\ell}_{-T-1} \equiv B^{\star} + \left(\tilde{B} - B^{\star} \right) \Gamma \left(T + 1 \right).$$

These are the sets of values of B that for which $B' \in \mathcal{B}_{-T+1}$, by using the Euler equation (27). Both sets have the property that, if we follow the Euler equation (27) and $B' \in \left[B^{\star}, \tilde{B} \right]$, for any $B \in \left[B^{\ell}_{-T}, B^{\ell}_{-T-1} \right]$, then $B' \in \mathcal{B}_{-T+1}$.

The next step is to continue backwards in time. At this point we work out the transition starting from debt in $\mathcal{B}_{-T}$ and $\tilde{\mathcal{B}}_{-T}$. The reason is debt levels in each interval, are reached using a different Euler equation at period $-T-1$. We characterize the sequence of sets $\mathcal{B}_{-T-\tau}$ and $\tilde{\mathcal{B}}_{-T-\tau}$ for $\tau \geq 1$.

Derivation of the sets $\mathcal{B}_{-T-\tau}$.   Fix a debt level $B_{-T} \in \mathcal{B}_{-T}$ and an associated consumption level $C \left(B_{-T} \right)$. This debt level is reached starting from a debt level $B_{-T-1} > \tilde{B}$, if it satisfies the budget equation:

$$qC_{-T-1} = C_{-T} \tag{34}$$

and

$$C_{-T-\tau} = C_{-T-\tau+1}, \quad \forall \tau \geq 1, \tag{35}$$

using the terminal condition, $C_{-T} = C \left(B_{-T} \right).$

Using the budget constraint, we have that:

$$C_{-T-1} = \frac{1}{q} \left(1 + \beta B_{-T} - B_{-T-1} \right),$$

so combined with (34) and the terminal condition, we obtain:

$$B_{-T-1} = 1 + \beta B_{-T} - C \left(B_{-T} \right), \quad \forall B_{-T} \in \mathcal{B}_{-T}.$$

We already showed, by using the Euler equation backwards, that $\tilde{B} < B^{\ell}_{-T-1} \leq B_{-T-1}$. Thus, for any $B_{-T} \in \mathcal{B}_{-T}$, we obtain a new value of debt $B_{-T-1} > \tilde{B}$. Hence,

I define an interval of terminal conditions:

$$B_{-T-1} = \{B : B = 1 + \beta B' - C(B'), \quad B' \in \mathcal{B}_{-T}\}.$$

I now use (35), backwards in time.

If $B_{-T-1} > \tilde{B}$, then B_{-T-1} can be reached if it satisfies the Euler equation:

$$C_{-T-2} = C_{-T-1} \rightarrow \frac{1}{q}(1 + \beta B_{-T-1} - B_{-T-2}) = \frac{1}{q}(1 + \beta B_{-T} - B_{-T-1}), \qquad (36)$$

where I used that if $B_{-T-1}, B_{-T-2} > \tilde{B}$ the worker's consumption is exclusively in chained goods.

Re-arranging terms in (36), we obtain:

$$B_{-T-2} = (1 + \beta) B_{-T-1} - \beta B_{-T}.$$

Generically, using (35):

$$C_{-T-\tau} = C_{-T-\tau-1} \rightarrow \frac{1}{q}(1 + \beta B_{-T-\tau-1} - B_{-T-\tau-2}) = \frac{1}{q}(1 + \beta B_{-T-\tau} - B_{-T-\tau-1}),$$

and likewise:

$$\begin{aligned} B_{-T-\tau-2} &= (1 + \beta) B_{-T-\tau-1} - \beta B_{-T-\tau} \\ &= B_{-T-\tau-1} + \beta (B_{-T-\tau-1} - B_{-T-\tau}). \end{aligned}$$

Then, subtracting $B_{-T-\tau+1}$ from both sides we arrive at:

$$B_{-T-\tau-2} - B_{-T-\tau-1} = \beta (B_{-T-\tau-1} - B_{-T-\tau}).$$

Thus, by iterating forward τ times:

$$B_{-T-\tau-2} - B_{-T-\tau-1} = \beta^{\tau+1} (B_{-T-1} - B_{-T}).$$

two periods forward:

$$B_{-T-\tau} - B_{-T-\tau+1} = \beta^{\tau-1} \left(B_{-T-1} - B_{-T} \right), \quad \forall \tau \geq 1.$$

Summing up all the differences up to τ, we obtain:

$$B_{-T-\tau} - B_{-T} = \sum_{s=1}^{\tau} \beta^{s-1} \left(B_{-T-1} - B_{-T} \right).$$

Setting $\tau \to \infty$, we obtain:

$$B_{-T-\tau} = B_{-T} + \frac{1}{1-\beta} \left(B_{-T-1} - B_{-T} \right).$$

Thus, the sets

$$
\begin{aligned}
\mathcal{B}_{-T-\tau} & \equiv \left\{ B_{-T-\tau} : B_{-T-\tau} = B_{-T} + \sum_{s=0}^{\tau-1} \beta^s \left(1 - C\left(B_{-T}\right) - \left(1-\beta\right) B_{-T} \right), \quad B_{-T} \in \mathcal{B}_{-T} \right\} \\
& = \left\{ B_{-T-\tau} : B_{-T-\tau} = \beta^\tau B_{-T} + \frac{1-\beta^\tau}{1-\beta} \left(1 - C\left(B_{-T}\right) \right), \quad B_{-T} \in \mathcal{B}_{-T} \right\},
\end{aligned}
$$

where I used $B_{-T-1} = 1 + \beta B_{-T} - qC\left(B_{-T-1}\right) = 1 + \beta B_{-T} - C\left(B_{-T}\right)$.

These sets characterize the set of value of $B_{-T-\tau}$ that reach a point B_{-T} in $\mathcal{B}_{-T}$, using the Euler equation. Taking the limit:

$$\lim_{t \to \infty} \mathcal{B}_{-T-\tau} \equiv \left\{ \frac{1}{1-\beta} \left(1 - C\left(B_{-T}\right) \right), \quad B_{-T} \in \mathcal{B}_{-T} \right\}.$$

We also have that:

$$\inf \mathcal{B}_{-T-\tau} = \beta^\tau B_{-T} + \frac{1-\beta^\tau}{1-\beta} \left(1 - \inf_{B_{-T} \in \mathcal{B}_{-T}} C\left(B_{-T}\right) \right) = \beta^\tau B_{-T} + \frac{1-\beta^\tau}{1-\beta} \left(1 - \frac{\left(\tilde{B} - B^\star \right)}{q^T} \right),$$

and

$$\inf \mathcal{B}_{-T-1} = B^\ell_{-T-1}.$$

Moreover:

$$\max \mathcal{B}_{-T-\tau} = \beta^\tau B_{-T} + \frac{1-\beta^\tau}{1-\beta}\left(1 - C\left(\tilde{B}\right)\right).$$

thus, showing the continuity of the interval.

Derivation of the sets $\tilde{\mathcal{B}}_{-T-\tau}$. Now consider the set, $\tilde{\mathcal{B}}_{-T}$. Fix a debt level $B_{-T} \in \tilde{\mathcal{B}}_{-T}$ and an associated consumption level $C\left(B_{-T}\right)$. This debt level is reached starting from a debt level $B_{-T-1} > \tilde{B}$, if it satisfies the budget equation:

$$C_{-T-\tau} = C_{-T-\tau+1}, \quad \forall \tau \geq 0, \tag{37}$$

using the terminal condition, $C_{-T} = C\left(B_{-T}\right).$

Using the budget constraint,

$$C_{-T-\tau} = \frac{1}{q}\left(1 + \beta B_{-T-\tau+1} - B_{-T-\tau}\right),$$

so combined with (37) we obtain:

$$\frac{1}{q}\left(1 + \beta B_{-T-\tau-1} - B_{-T-\tau-2}\right) = \frac{1}{q}\left(1 + \beta B_{-T-\tau} - B_{-T-\tau-1}\right),$$

and likewise:

$$B_{-T-\tau-2} = B_{-T-\tau-1} + \beta\left(B_{-T-\tau-1} - B_{-T-\tau}\right).$$

Then, subtracting $B_{-T-\tau+1}$ from both sides we arrive at:

$$B_{-T-\tau-2} - B_{-T-\tau-1} = \beta\left(B_{-T-\tau-1} - B_{-T-\tau}\right).$$

Thus, by iterating forward τ times:

$$B_{-T-\tau-2} - B_{-T-\tau-1} = \beta^{\tau+1}\left(B_{-T-1} - B_{-T}\right).$$

By change of variables:

$$B_{-T-\tau} - B_{-T-\tau+1} = \beta^{\tau-1} \left(B_{-T-1} - B_{-T} \right), \quad \forall \tau \geq 1.$$

where

$$B_{-T-1} = 1 + \beta B_{-T} - qC \left(B_{-T} \right), \quad \forall B_{-T} \in \tilde{\mathcal{B}}_{-T}.$$

Summing up all the differences up to τ, we obtain:

$$B_{-T-\tau} - B_{-T} = \sum_{s=1}^{\tau} \beta^{s-1} \left(B_{-T-1} - B_{-T} \right) = \sum_{s=0}^{\tau-1} \beta^s \left(B_{-T-1} - B_{-T} \right).$$

Solving the summation term:

$$B_{-T-\tau} = B_{-T} + \frac{1 - \beta^{\tau}}{1 - \beta} \left(B_{-T-1} - B_{-T} \right).$$

Thus, we obtain the sets:

$$
\begin{aligned}
\tilde{\mathcal{B}}_{-T-\tau} &\equiv \left\{ B_{-T-\tau} : B_{-T} + \sum_{s=0}^{\tau-1} \beta^s \left(1 - qC \left(B_{-T} \right) - \left(1 - \beta \right) B_{-T} \right), \quad B_{-T} \in \tilde{\mathcal{B}}_{-T} \right\}, \\
&= \left\{ B_{-T-\tau} : \beta^{\tau} B_{-T} + \frac{1 - \beta^{\tau}}{1 - \beta} \left(1 - qC \left(B_{-T} \right) \right), \quad B_{-T} \in \tilde{\mathcal{B}}_{-T} \right\}.
\end{aligned}
$$

where I used that $B_{-T-1} = 1 + \beta B_{-T} - qC \left(B_{-T} \right)$.

Setting $\tau \to \infty$, we obtain:These sets characterize the set of value of $B_{-T-\tau}$ that reach a point B_{-T} in $\tilde{\mathcal{B}}_{-T}$, using the Euler equation. Taking the limit:

$$\lim_{t \to \infty} \tilde{\mathcal{B}}_{-T-\tau} \equiv \left\{ B_{-T-\tau} : \frac{1}{1 - \beta} \left(1 - qC \left(B_{-T} \right) \right), \quad B_{-T} \in \tilde{\mathcal{B}}_{-T} \right\}.$$

We also have that:

$$
\begin{aligned}
\max \tilde{\mathcal{B}}_{-T-\tau} &= \beta^{\tau} B_{-T} + \frac{1 - \beta^{\tau}}{1 - \beta} \left(1 - q \max_{B_{-T} \in \tilde{\mathcal{B}}_{-T}} C \left(B_{-T} \right) \right) \\
&= \beta^{\tau} B_{-T} + \frac{1 - \beta^{\tau}}{1 - \beta} \left(1 - \frac{\left(\tilde{B} - B^{\star} \right)}{q^T} \right),
\end{aligned}
$$

and, thus,

$$\max \tilde{\mathcal{B}}_{-T-1} = B^\ell_{-T-1}.$$

Moreover, we have that:

$$\inf \tilde{\mathcal{B}}_{-T-\tau} = \beta^\tau B_{-T} + \frac{1-\beta^\tau}{1-\beta}\left(1 - q \inf_{B_{-T} \in \tilde{\mathcal{B}}_{-T}} C\left(B_{-T}\right)\right) = \beta^\tau B_{-T} + \frac{1-\beta^\tau}{1-\beta}\left(1 - qC\left(\tilde{B}\right)\right).$$

Thus, all sets $\tilde{\mathcal{B}}_{-T-\tau}$ and $\mathcal{B}_{-T-\tau}$ overlap.

$\square$

We are now ready to proof the statements of the Proposition.

Proof of Part II.a.   I now proceed to proof Part II of the Proposition. I begin by show-ing the following Lemma regarding the optimal policies for initial values of debt, $B_0 \in (B^\star, \tilde{B}]$:

Lemma 11. *For any $B \in (B^\star, \tilde{B}]$, we have have $V(B) > \underline{V}(B)$, $B' \leq B$ and $B_t \to B^\star$ where the convergence occurs in at most $T + 1$ periods.*

Proof. The proof proceeds in three steps. First, using the Euler equation, I show that the value function satisfies $V(B) > \underline{V}(B)$. Second, I show that if the Euler equation holds, debt cannot exit the interval exceeding the upper bound. Finally, debt must exit the interval from below.

Step 1. Consider first the interval $B \in \left[B^\star\left(\tilde{B}\right), \tilde{B}\right]$. By Proposition 4, in the main text we have that:

$$\frac{C\left(B'\right)}{C\left(B\right)} = \beta R \frac{q^E\left(B', \tilde{B}\right)}{q^B\left(B', \tilde{B}\right)} = \frac{1 + (q-1)\,\mathbb{I}_{\left[B' \geq B^\star(\tilde{B})\right]}}{1 + (q-1)\,\mathbb{I}_{\left[B' > \tilde{B}\right]}} = q > 1. \tag{38}$$

Clearly, if $E^w = 1 - (1-\beta)B$ then $B' = B$ and $C(B) = C(B')$, which contradicts the condition above, (38). Since $B' = \tilde{B}$ is a sub-optimal debt policy in $B \in \left[B^\star\left(\tilde{B}\right), \tilde{B}\right]$ that coincided with the optimal policy of $\underline{V}\left(\tilde{B}\right)$, then $V\left(\tilde{B}\right) > \underline{V}\left(\tilde{B}\right)$.

Step 2. Next, I verify that if $B \in (B^\star, \tilde{B}]$, then B must exit the interval (from below) at some finite time. Assume, by contradiction, that B_t never exits the interval. Then, by

(38), we would have that:

$$\lim_{t \to \infty} \frac{C_t}{C_0} = \lim_{t \to \infty} q^t = \infty,$$

hence, a contradiction. Thus, there must exist a finite time T after which B exits $\left[B^\star, \tilde{B} \right]$.

Suppose, by false assumption, that B_t exits to the right of $\tilde{B}$. Then, there exists a largest finite time T such that B remains in the interval and B, but exits from above at $T+1$. If debt exits the interval from above, from $T+1$ onwards, consumption and debt must be constant:

$$C_t = C_{t+1} \text{ and } B_t = B_{t+1} \quad t \geq T.$$

Constant consumption must hold, because the Euler equation for $B > \tilde{B}$ requires so. Constant consumption does not imply constant debt in general. However, a constant debt path must follow because debt increases at T, so is not feasible to have constant consumption and return to the interval (given higher average prices for larger debt). Thus, either debt explodes (which is unfeasible) or debt must be constant.

Constant debt requires:

$$E^w_{T+1} = 1 - (1 - \beta) B_{T+1}. \tag{39}$$

From the arguments above, there must exist an optimal sequence of consumption such that:

$$C_0 < C_1 = qC_0 < C_2 = q^2 C_0 \ldots < C_T = q^T C_0 = C_{T+1} = C_{T+2} = C_{T+3}...$$

Since by, false assumption, the sequence of debt is increasing, it must be that:

$$E^w_0 > 1 - (1 - \beta) B_0.$$

which implies

$$E^w_{T+1} > q^T E^w_0 > E^w_0 > 1 - (1 - \beta) B_0, \tag{40}$$

where the first inequality comes from:

$$E^w_{T+1} = Q_{T+1} C_{T+1} = Q_{T+1} q^T C_0 = \frac{Q_{T+1}}{Q_0} q^T E^w_0,$$

where $Q_{T+1} > Q_0$ (spot consumption is lower for higher debt levels if overall consumption is fixed). However, combining (39) and (40), we have that $B_{T+1} < B_0$ which contradicts the stated assumption that B_t exits the interval from above.

Step 3. Since B_t does not exit the interval exceeding the upper bound, it must exit the interval from below. Moreover, debt remains in the interval a finite amount of periods, T. Furthermore, we have that for $B_t = B_{T+1}$, for $t \geq T + 1$. There are two possibilities, either

$$B_{T+1} = B^\star \text{ or } B_{T+1} < B^\star.$$

Consider Proposition 4. Assume $B_{T+1} < B^\star$. Then, the Euler condition is:

$$\frac{E_{T+1}^w}{E_T^w} \frac{Q_T}{Q_{T+1}} = \beta \frac{R_{t+1}}{\Pi_{T+1}(B_{T+1})} \rightarrow \frac{E_{t+1}^w}{E_t^w} = \frac{1}{Q_T} < 1,$$

where I used the definition of Π_{t+1}:

$$\Pi_{T+1}(B_{T+1}) \equiv \tilde{q}_{T+1}^B(B_{T+1}) / \tilde{q}_T^E(B_{T+1}) = \frac{1 + (q-1)\,\mathbb{I}_{\left[B' > \tilde{B}_{ss}\right]}}{1 + (q-1)\,\mathbb{I}_{\left[B' \geq B^\star\left(\tilde{B}_{ss}\right)\right]}} = 1.$$

and that $Q_t > Q_{T+1} = 1$. However, this leads to a contradiction because:

$$\frac{E_{t+1}^w}{E_t^w} = \frac{1 - (1-\beta)\,B_{T+1}}{1 - B_T + \beta B_{T+1}} > 1 \text{ if } B_{T+1} < B_T.$$

Thus, the only possibility is that B' exits the interval reaching $B^\star\left(\tilde{B}\right)$ in finite time. For this, we must verify the inequality in Proposition 4. By Proposition 11, this condition is verified if $B_T \in \mathcal{B}_0$. If in turn, $B_T \notin \mathcal{B}_0$, then T is not final exit time. Instead, according to Proposition 12, there exists a sequence of B_t that exits the interval reaching $B^\star$ in $t > T$. In either case, it must be the case that B_t exits from below and reaches $B^\star$, as stated by the Proposition.

$\square$

Proof of Part II.b.   I now proceed to proof Part II.b of the Proposition. First we show the following corollary to Proposition 13.

Corollary 5. *There exists a maximal debt level*

$$\underline{B} = \frac{1 - q^{-T}\left(1 - (1 - \beta)\, B^\star\right)}{1 - \beta},$$

such that deleveraging toward $B^\star$, while following the Euler equation (27) is not feasible.

Proof. Recall that $\tilde{\mathcal{B}}_{-T-\tau} \cup \mathcal{B}_{-T-\tau}$ represents the set of values where there is a deleveraging path such that debt reaches $B^\star$ in $T + \tau + 1$ periods. In all cases, the Euler equation (38) is satisfied. From (33), we have that

$$\underline{B} \equiv \lim_{\tau \to \infty} \tilde{\mathcal{B}}_{-T-\tau} \cup \mathcal{B}_{-T-\tau} = \frac{1 - q^{-T}}{1 - \beta}\left(1 - (1 - \beta)\, B^\star\right).$$

The debt threshold $\underline{B}$ defines a maximally high debt level for which deleveraging in finite time is not possible while respecting the Euler equations.

$\square$

Existence of B^h. Thus, from this Corollary we have that for values of debt above the threshold $\underline{B}$, the policy $B' = B$, is the only solution consistent with the Euler equation. We have:

Corollary 6. *For any $B \geq \underline{B}$, we have $V(B) = \underline{V}(B)$.*

Thus, we know that $V\left(\tilde{B}\right) > \underline{B}\left(\tilde{B}\right)$ while $V(\underline{B}) = \underline{V}(\underline{B})$. By continuity of V, there exists some $B^h \in \left[\tilde{B}, \underline{B}\right]$ such that $V(B) = \underline{V}(B)$, for any $B > B^h$.

Uniqueness of B^h. Next, we show uniqueness. To do so, I will show that if $V(B^d) > \underline{V}(B^d)$, then $V(B) > \underline{V}(B)$ for any $B < B^d$, a condition that guarantees the uniqueness of the threshold B^h.

Fix any $B^d > \tilde{B}$ such that $V(B^d)¿\underline{V}(B^d)$. We know this B^d exists because $V\left(\tilde{B}\right) > \underline{V}(B^d)$ and the value function is continuous. If this condition holds, then $B^d \in \tilde{\mathcal{B}}_{-T-\tau} \cup \mathcal{B}_{-T-\tau}$ for some finite $\tau \geq 1$. Then, using the pattern of convergence, we can opening

up the value and using the results from (33), we obtain:

$$
\begin{aligned}
V\left(B^d\right) &= \frac{1-\beta^{\tau+1}}{1-\beta}\ln\left(\frac{1}{q^{T+1}}C\left(B_{-T}\right)\right) + \beta^\tau\sum_{s=1}^{T}\beta^s\ln\left(\frac{q^s}{q^{T+1}}C\left(B_{-T}\right)\right)\cdots \quad (41)\\
&+\ \beta^{T+\tau+1}\frac{\ln\left(1-(1-\beta)B^\star\right)}{1-\beta}\cdots\\
&>\ \frac{\ln\left(\frac{1-(1-\beta)B^d}{q}\right)}{1-\beta} = \underline{V}\left(B^d\right).
\end{aligned}
$$

for some sequence such that $B_{-t}\in\mathcal{B}_{-t}$. If we multiply both sides by $(1-\beta)$ and subtract the left from the right, we obtain that $(1-\beta)V\left(B^d\right)-(1-\beta)\underline{V}\left(B^d\right)$ is of the form:

$$
G\left(\Delta\right)\equiv\sum_{s=0}^{\infty}\left(1-\beta\right)\beta^\tau\left(\ln\left(x_s+\Delta\right)-\ln\left(\underline{x}+\Delta\right)\right).
$$

for some pair of bounded sequences $\{x_s\}_{s\geq0}$ and constants $\underline{x}$, and evaluate at $\Delta = 0$. Since $(1-\beta)V\left(B^d\right)-(1-\beta)\underline{V}\left(B^d\right)>0$, we have that $G\left(0\right)>0$. Then, since $\ln$ exhibits decreasing absolute risk-aversion, and the sequence $(1-\beta)\beta^\tau$ corresponds to a weighted sum that adds to one, we thus have that $G\left(\Delta\right)>0$, for any $\Delta>0$.

Now fix any $B<B^d$ and set $\Delta\equiv B^d-B$. We have that:

$$
\underline{V}(B) = \frac{\ln\left(\frac{1-(1-\beta)B^d+\Delta(1-\beta)}{q}\right)}{1-\beta} = \sum_{s=0}^{\infty}\beta^\tau\ln\left(\frac{1-(1-\beta)B^d+\Delta(1-\beta)}{q}\right).
$$

We know that starting from B^d, the worker follows a sequence B_{-t} that converges to $B^\star$ by $t=1$. Then, consider the problem starting from $B<B^d$. Starting from such B, a feasible policy is to consume the annuity of Δ in chained goods forever, and follow the same deleveraging path for $\{B_{-t}\}$ as the path that delevers to $B^\star$ starting from B^d

(excluding the annuity):

$$(1-\beta)\,V\left(B^d-\Delta\right) \quad > \quad (1-\beta)\sum_{s=0}^{\tau}\beta^\tau \ln\left(\frac{1}{q^{T+1}}C\left(B_{-T}\right)+\frac{\Delta\left(1-\beta\right)}{q}\right)\cdots$$

$$+ \quad (1-\beta)\,\beta^\tau \sum_{s=1}^{T}\beta^s \ln\left(\frac{q^s}{q^{T+1}}C\left(B_{-T}\right)+\frac{\Delta\left(1-\beta\right)}{q}\right)$$

$$+(1-\beta)\sum_{s=T+\tau+1}^{\infty}\beta^s \ln\left(1-(1-\beta)\,B^\star+\frac{\Delta\left(1-\beta\right)}{q}\right).$$

The inequality follows because the sequence in the right is sub-optimal. However, the expression to the right is also of the form taken by $G\left(\Delta\right)$, hence we have by decreasing absolute risk aversion that:

$$(1-\beta)\,V\left(B^d-\Delta\right)-(1-\beta)\,\underline{V}\left(B^d-\Delta\right)>0, \tag{42}$$

for any Δ.

Equation (42) guarantees that there exists a unique minimum threshold value B^h for which $V\left(B^h\right)=\underline{V}\left(B^h\right)$.[28] This completes the proof of Proposition 5.

B.4 Description: Complete Characterization of the Worker's Problem

In Section 3 I present Proposition 5 and a numerical solution to the worker's problem, in Figure 3. In this appendix, I show a more general version of Proposition 5, based on the results derived above. If we combine Propositions 12 and 13, we know that for any $B\in\left(B^\star,B^h\right)$, debt converges to $B^\star$ in finite time. Working backwards in time, and setting $t=1$ the first periods such that $B=B^\star$, the propositions show that:

$$B_{-t}\in\mathcal{B}_{-t}$$

---

[28]Suppose not. Then, since $V\left(B\right)$ is continuous, it would cross $\underline{V}$ twice.

for $t \in \{0, 1, ..., T\}$ and

$$B_{-t} \in \mathcal{B}_{-t} \cup \tilde{\mathcal{B}}_{-t},$$

for $t > T$.

In the proofs, I characterized the left boundaries of the intervals $\mathcal{B}_{-t}$, the terms B^ℓ_{-t} and $\tilde{B}^\ell_{-t}$. Figure 7 reproduces Figure 3 but includes the values of B^ℓ_{-t} and $\tilde{B}^\ell_{-t}$ and the corresponding values of $V(B)$, evaluated at such points—which are given by equation (41). We should observe multiple features consistent with the propositions in the proof. The red and green vertical bars care located at the values of $\left\{ B^\ell_{-t}, \tilde{B}^\ell_{-t} \right\}$. The red and green scatter points correspond to the values of $\left\{ B^\ell_{-t}, V\left(B^\ell_{-t} \right) \right\}$ and $\left\{ \tilde{B}^\ell_{-t}, V\left(\tilde{B}^\ell_{-t} \right) \right\}$ respectively.

1. The sequences B^ℓ_{-t} and $\tilde{B}^\ell_{-t}$ converge to a constant for $t \to \infty$.

2. The value function $V(B)$ is above the values given by the scatter points.

3. The point B^h is located near the crossing point of where $\underline{V}\left(B^\ell_{-t} \right)$ and the scatter points.

B.5 Proof of Proposition 6

For convenience, I reproduce the aggregate Euler equation (15) here:

$$\underbrace{\frac{B}{1-(1-\beta)B} \cdot Q\left(B, \tilde{B}\right)}_{\equiv \mathcal{E}\left(B; \tilde{B}\right)} = \underbrace{\frac{B'}{1-(1-\beta)B'} \frac{Q\left(B', \tilde{B}'\right)}{\Pi\left(B'; B, \tilde{B}, \tilde{B}'\right)}}_{\equiv \mathcal{E}'\left(B'; B, \tilde{B}, \tilde{B}'\right)}.$$

First, I show that for a subset of $B > \tilde{B}'$ there are two roots B' that solve $\mathcal{E} = \mathcal{E}'$. The roots satisfy $B'_1 < \tilde{B}' < B'_2$ and $B'_2 = B$ thus $R = \beta^{-1}$ and $q = q'$. Then, the proof finishes by show that do to the hysteresis result of stationary problems, for each $B \in \left(\tilde{B}', B^h \right)$ the larger root B'_2 cannot be an individual optimum during a transition.

The first result is summarized by the following Lemma.

Lemma 12. *There exists a threshold B^* such that for any $B \in [\tilde{B}', B^*]$, the equation $\mathcal{E} = \mathcal{E}'$ has two roots B': one root is $B = B'$ and the other satisfies $B' \leq \tilde{B}'$. For any*

(a) Value function

(b) Debt Policy

(c) Consumption Policy
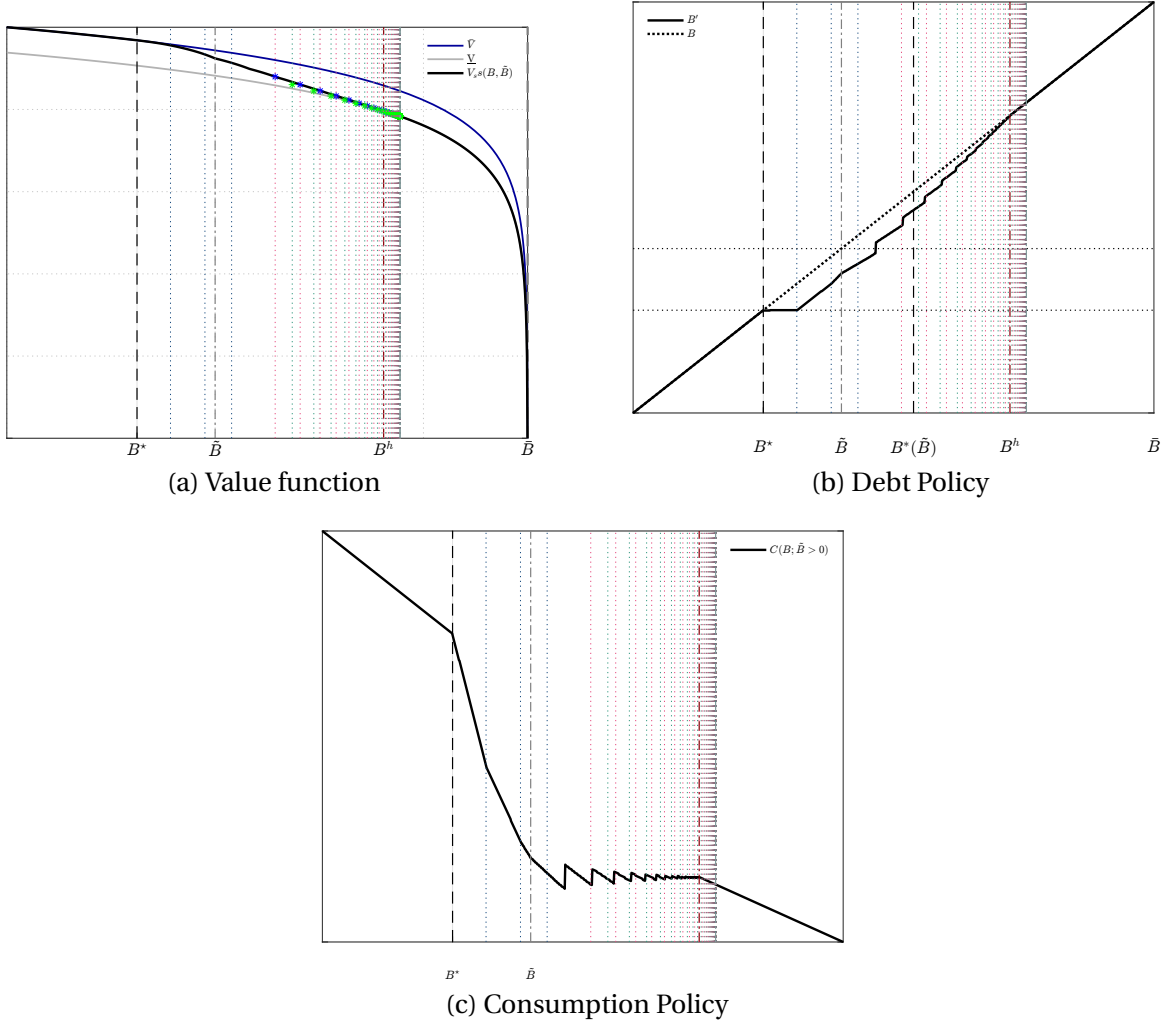
Figure 7: Bellman Equation and Policy Functions: Worker Problem

Note: Figures are calculated using value function iteration: $\beta = 0.8$, $q = 1.75$ and $\tilde{B} = 0.4 \cdot \bar{B}$.

$B > B^*$, only $B = B'$ is a solution. B^* solves:

$$\mathcal{E}'\left(B^*; \tilde{B}, \tilde{B}, \tilde{B}'\right) = \lim_{B' \uparrow \tilde{B}'} \mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right).$$

The interpretation of B^* is obtained from the left panel of Figure 4. B^* is the highest value for which the Euler equation has a solution $B' \in (\tilde{B}', B^*)$.

Proof. Since $\tilde{B}' > \tilde{B}$, if $B' = B > \tilde{B}'$, the worker only spends chained expenditures.

Thus,

$$\Pi\left(B'; B, \tilde{B}, \tilde{B}'\right) = 1 \text{ and } Q\left(B, \tilde{B}\right) = Q\left(B', \tilde{B}'\right) = q\left(1 - (1-\beta)B\right).$$

Hence, $B' = B$ is always a solution to the Euler equation when $B' = B > \tilde{B}'$.

Let $B > B^*$. The function $\mathcal{E}(B)$ is monotone increasing. In turn, the function $\mathcal{E}'(B')$ is monotone increasing to the left of $\tilde{B}'$. Hence, for $B > B^*$ a solution must lie at $B' > \tilde{B}'$. However, for any $B' > \tilde{B}'$,

$$\mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right) = \frac{B'}{1 - (1-\beta)B'} q\left(1 - (1-\beta)B'\right)$$

which follows from the fact that $B' > \tilde{B}' > B^\star\left(1/\beta, \tilde{B}'\right)$ in which case $\Pi\left(B'; B, \tilde{B}, \tilde{B}'\right) = 1$. This implies that for any $B > B^*$ we have that:

$$\mathcal{E}\left(B; \tilde{B}\right) = \mathcal{E}'\left(B'; B, \tilde{B}, \tilde{B}'\right),$$

which implies,

$$\frac{B}{1 - (1-\beta)B} \cdot q\left(1 - (1-\beta)B\right) = \frac{B'}{1 - (1-\beta)B'} \cdot q\left(1 - (1-\beta)B'\right).$$

By monotonicity, we immediately conclude that $B = B'$ and that is the only solution.

Now consider $B \in \left(\tilde{B}', B^*\right)$. We now look for a solution $B' \in \left(B^\star(\tilde{B}'), \tilde{B}'\right)$. If $B < B^*$, again by monotonicity:

$$\mathcal{E}'\left(B; \tilde{B}, \tilde{B}, \tilde{B}'\right) < \lim_{B'\uparrow\tilde{B}'} \mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right).$$

$$\mathcal{E}\left(B; \tilde{B}\right) = \mathcal{E}'\left(B'; B, \tilde{B}, \tilde{B}'\right)$$

$$\mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right) = \lim_{B' \downarrow \tilde{B}'} \mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right)$$

$$\frac{B'}{1 - (1 - \beta) B'} \cdot Q\left(B', \tilde{B}'\right) = \frac{B'}{1 - (1 - \beta) B'} \cdot \frac{q\left(1 - (1 - \beta) \tilde{B}'\right)}{\lim_{B' \uparrow \tilde{B}'} \Pi\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right)}$$

$$\frac{B'}{1 - (1 - \beta) B'} \cdot Q\left(B', \tilde{B}'\right) = \frac{\tilde{B}'}{1 - (1 - \beta) \tilde{B}'} \cdot \frac{q\left(1 - (1 - \beta) \tilde{B}'\right)}{\frac{q\left(1 - (1 - \beta) \tilde{B}'\right)}{q\left(1 - (1 - \beta) \tilde{B}\right)}}$$

$$\frac{B'}{1 - (1 - \beta) B'} \cdot Q\left(B', \tilde{B}'\right) = \frac{\tilde{B}'}{1 - (1 - \beta) \tilde{B}'} \cdot q\left(1 - (1 - \beta) \tilde{B}\right) > 0$$

Fix $\tilde{B}, \tilde{B}'$. Define $B^* > \tilde{B}$ to be the value that solves:

$$\mathcal{E}'\left(B^*; \tilde{B}, \tilde{B}, \tilde{B}'\right) = \lim_{B' \uparrow \tilde{B}'} \mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right).$$

By definition, if indeed $B^* > \tilde{B}$, then

$$\frac{B^*}{1 - (1 - \beta) B^*} \cdot q\left(1 - (1 - \beta) B^*\right) = \lim_{B' \uparrow \tilde{B}'} \frac{B'}{1 - (1 - \beta) B'} \cdot \frac{Q\left(B', \tilde{B}'\right)}{\Pi\left(B'; B^*, \tilde{B}, \tilde{B}'\right)}$$

which implies

$$B^* = \frac{\tilde{B}'}{\frac{1 - (1 - \beta) \tilde{B}'}{q\left(1 - (1 - \beta) \tilde{B}'\right)} + (1 - \beta) \tilde{B}'} = \frac{\tilde{B}'}{C^w\left(\tilde{B}', \tilde{B}'\right) + C^s\left(\tilde{B}'\right)} > \tilde{B}'.$$

The last inequality verifies that $B^* > \tilde{B}'$. This result follows from

$$\lim_{B' \uparrow \tilde{B}'} \Pi\left(B'; B^*, \tilde{B}, \tilde{B}'\right) = \left(q\left(1 - (1 - \beta) B^*\right)\right)^{-1}$$

and

$$\lim_{B' \uparrow \tilde{B}'} Q\left(B', \tilde{B}'\right) = q\left(1 - (1 - \beta) \tilde{B}'\right).$$

The inequality holds since $q > 1$. Furthermore, this satisfies $B^* > \tilde{B}$ as required since

the SBL sequence is weakly increasing.[29]

Next, I show that for any $B \in \left(\tilde{B}', B^* \right)$, the equation $\mathcal{E} = \mathcal{E}'$ has two roots; one above $\tilde{B}'$ and one below. The solution above $\tilde{B}'$ is trivial: $B = B' > \tilde{B}' \geq \tilde{B}$ satisfies the equation. For the solution below, I use continuity and monotonicity to show that there is a unique $B' < \tilde{B}'$ such that

$$\mathcal{E}' \left(B'; \tilde{B}, \tilde{B}, \tilde{B}' \right) = \lim_{B' \downarrow \tilde{B}'} \mathcal{E}' \left(B'; \tilde{B}, \tilde{B}, \tilde{B}' \right)$$

$$\frac{B'}{1 - (1 - \beta) B'} \cdot Q \left(B', \tilde{B}' \right) = \frac{B'}{1 - (1 - \beta) B'} \cdot \frac{q \left(1 - (1 - \beta) \tilde{B}' \right)}{\lim_{B' \uparrow \tilde{B}'} \Pi \left(B'; \tilde{B}, \tilde{B}, \tilde{B}' \right)}$$

$$\frac{B'}{1 - (1 - \beta) B'} \cdot Q \left(B', \tilde{B}' \right) = \frac{\tilde{B}'}{1 - (1 - \beta) \tilde{B}'} \cdot \frac{q \left(1 - (1 - \beta) \tilde{B}' \right)}{\frac{q \left(1 - (1 - \beta) \tilde{B}' \right)}{q \left(1 - (1 - \beta) \tilde{B} \right)}}$$

$$\frac{B'}{1 - (1 - \beta) B'} \cdot Q \left(B', \tilde{B}' \right) = \frac{\tilde{B}'}{1 - (1 - \beta) \tilde{B}'} \cdot q \left(1 - (1 - \beta) \tilde{B} \right) > 0$$

Lets call this B', $\underline{B}$ and think of the interval $\left(\tilde{B}', B^* \right)$. The interpretation of $\underline{B}$ is that it is the small root for the debt level "$\tilde{B}' + \varepsilon$". Meaning, if $\tilde{B}'$ is the small root for B^*(the end of the interval) then $\underline{B}$ is the small root for the start of the interval. Now, notice that the RHS is a constant with respect to $\tilde{B}'$. To establish existence of $\underline{B}$ further notice that the LHS tends to $\lim_{B' \uparrow \tilde{B}'} \mathcal{E}' \left(B'; B^*, \tilde{B}, \tilde{B}' \right)$ as $B' \uparrow \tilde{B}'$ (this was shown in the previous step) and this is larger than the RHS just by comparing magnitudes ($\tilde{B}' \geq \tilde{B}$ and $q \left(1 - (1 - \beta) B^* \right) > 1$). Then notice that as $B' \downarrow 0$ the LHS goes to zero which is lower than the RHS. By continuity of the LHS we can apply the intermediate value theorem for existence. Uniqueness is granted since the LHS is increasing in B'. The fraction is clearly increasing in B', the average price too because as $B' \uparrow \tilde{B}'$ the share of chained expenditure increases and also its price does so. This statement also uses the fact that Q is the weighted harmonic mean of prices with expenditure weights. Since the LHS is

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[29]This is important because the relevant points to study are those for which the all chained equilibrium is possible. These are the cases where $B^* > \tilde{B}' \geq \tilde{B}$.

continuous and increasing it maps the interval $\left(\underline{B}, \tilde{B}'\right)$ onto

$$\left(\lim_{B' \downarrow \tilde{B}'} \mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right), \lim_{B' \uparrow B^*} \mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right)\right).$$

So (since $\mathcal{E}'$ is increasing and continuous) it covers all the image of $\left(\tilde{B}', B^*\right)$. This proves that for $B \in \left(\tilde{B}', B^*\right)$ there exist two roots that solve $\mathcal{E} = \mathcal{E}'$. One is $B'_2 = B > \tilde{B}'$ and the other is a $B'_1 \in \left(\underline{B}, \tilde{B}'\right)$.

To make affairs clearer, suppose that $B^* < B^h$ then for all $B \in \left(\tilde{B}', B^*\right)$ the larger root $B' = B$ is not an equilibrium and there is a region (B^*, B^h) that does not have a symmetric competitive equilibrium because the only root that solves equation (15) provides allocations and prices that are not an equilibrium. Now suppose that $B^* > B^h$, then for $B \in \left(\tilde{B}', B^h\right)$ the larger root $B' = B$ is not an equilibrium and for $B \in (B^h, \overline{B})$. So summarizing both cases. For $B_0 < B^h\left(\tilde{B}_0\right)$, if a (symmetric competitive) equilibrium exists then it is given by the smaller root of equation 15.

□

B.6   Proof of Corollary 1

At $B_t \leq B^\star\left(\beta^{-1}, \tilde{B}_{ss}\right)$ if $R = \beta^{-1}$ then marginal and average prices are equal to 1. Then it is evident that $B_{t+1} = B_t$ solves equation 15 and the expenditure is $1 - (1 - \beta) B_t \leq \tilde{B}_{ss} - B_t$ by assumption. As a consequence, the steady state is non-disrupted.

B.7   Proof of Corollary 2

That $B_{t+1} < B_t$ if $B_t \in \left(\tilde{B}, B^*\left(\tilde{B}\right)\right)$ is immediate from the result proved in Proposition 6 since we are choosing the smaller root and the larger root is $B_{t+1} = B_t$. For $B_t > B^\star$ it is enough to show that $\beta R_{t+1} < 1$. This was done in step 3 of the proof of Proposition 6.

B.8   Proof of Corollary 3

The proof follows immediately from Proposition 5. If $B_0 > B^h$, there exists an equilibrium with $R_t = 1/\beta$ and $B_t = B_0$. Recall that if $R_t = 1/\beta$, and $B_t = B_{t+1}$ is a solution to $\mathcal{E}_t = \mathcal{E}_{t+1}$. Since for $R_t = 1/\beta$, $B_0 = B_t$ is a solution to the worker's problem, both Euler equations are satisfied and the asset clearing condition holds, sequences with $R_t = 1/\beta$ and $B_t = B_0$ are consistent with the equilibrium definition. Since at $B_t > B^h > B^\star$, $q_t > 1$. Hence, the economy remains permanently disrupted.

C.   Proofs of Section 4

C.1   Proof of Proposition 7

The strategy of the proof is to show that any solution to the Ramsey Problem satisfies the constraints in the Primal Problem (step 1), that solutions respect the optimal expenditure rules, and, finally, that any solution to the Primal Problem can be induced by a proper tax sequence $\{\tau_{t+1}^k, \tau_t^c, \tau_{t+1}^\ell\}_{t\geq 0}$ (step 2).

Step 1. The constraint set in Primal Problem contains constraints in Ramsey Problem.   Take the household budget constraints and the government budget in the original Ramsey Problem:

$$\left(1 + \tau_{t+1}^k\right) \frac{B_{t+1}}{R_{t+1}} + (1 + \tau_t^c) C_t^s = B_t, \ \forall t \geq 0$$

$$B_t + (1 + \tau_t^c) \left(S_t^w + q_t X_t^w\right) = \frac{B_{t+1}}{R_{t+1}} + 1 - \tau_{t+1}^\ell, \ \forall t \geq 0$$

and

$$\tau_{t+1}^k \frac{B_{t+1}}{R_{t+1}} + \tau_t^c \left(C_t^s + C_t^w\right) + \tau_{t+1}^\ell = 0, \ \forall t \geq 0.$$

If we add the first two constraints and cancel common terms, we obtain:

$$(1 + \tau_t^c) \left(S_t^w + q_t X_t^w\right) + \tau_{t+1}^k \frac{B_{t+1}}{R_{t+1}} + (1 + \tau_t^c) C_t^s = 1 - \tau_{t+1}^\ell.$$

If we then subtract the government budget constraint from this last equation, we obtain:

$$(1 + \tau_t^c) \left(S_t^w + q_t X_t^w\right) + \tau_{t+1}^k \frac{B_{t+1}}{R_{t+1}} + (1 + \tau_t^c) C_t^s - \left(\tau_{t+1}^k \frac{B_{t+1}}{R_{t+1}} + \tau_t^c \left(C_t^s + S_t^w + q_t X_t^w\right) + \tau_t^\ell\right) = 1 - \tau_{t+1}^\ell.$$

Cancelling terms, this condition further becomes:

$$S_t^w + q_t X_t^w + C_t^s = 1. \tag{43}$$

Finally, using $q_t = \mathcal{A}(\mu_t)^{-1}$, we obtain:

$$S_t^w + \mathcal{A}(\mu_t)^{-1} X_t^w + C_t^s = 1 \text{ and } \mu_t = \mathcal{A}(\mu_t)^{-1} X_t^w.$$

This shows that any solution to the Ramsey Problem satisfies the constraints of the Primal Problem.

Step 2. The Ramsey Planner can implement the Primal Problem Solution.   Next, observe that for any choice of C_t^w in the Primal Problem, the Primal Planner is better off maximizing S_t^w since $\mathcal{A}(\mu_t)^{-1} \geq 1$. Hence, it must be the case that:

$$S_t^w = \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, 1 - C_t^s\right\}.$$

Then, by definition:

$$X_t^w = E_t^w - \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, 1 - C_t^s\right\}/q_t.$$

In this expression, I used that $E_t^w = 1 - C_t^s$ exploiting the expenditure-income—equation (43). Since both the Ramsey and the Primal problems induce the same level of consumption for workers given a level of saver consumption, the value of both problems coincides if they can induce the same set of saver consumption paths.

In the primal problem, I use that

$$C_t^s = (1 - \beta) B_t.$$

Thus, since the planner in the Primal Problem can choose the path of debt directly, it can chose saver expenditures as well. Since the constraint set in the Primal is a subset of the constraint in the Ramsey problem, the primal problem is more relaxed than the original Ramsey problem. Hence, If the Ramsey planner can achieve the same level of saver expenditures as the Primal planner, then, it must achieve the same value. The next Lemma is used to verify that claim.

Lemma 13. *Let* $\left\{\tau_{t+1}^k, \tau_t^c, \tau_{t+1}^\ell\right\}_{t \geq 0}$ *be a sequence of taxes in the Ramsey Problem. The*

solution to the saver's problem is given by:

$$C_t^s = (1 - \beta) B_t^o$$

where

$$B_0^o = \frac{B_0}{(1 + \tau_0^c)}$$

$$B_{t+1}^o = \hat{R}_{t+k} \beta B_t^o$$

and

$$\hat{R}_{t+k} \equiv \frac{R_{t+1}}{\left(1 + \tau_{t+1}^k\right)} \frac{1}{\left(1 + \tau_{t+1}^c\right) / \left(1 + \tau_t^c\right)}.$$

This Lemma is the solution to the saver's problem. The Lemma implies that any sequence of solutions to the Primal Problem can be reproduced by the the Ramsey Planner.

Indeed, let $\{B_t^o\}_{t \geq 0}$ be a solution to the primal problem. Then, the Ramsey planner can set

$$(1 + \tau_0^c) = B_0 / B_0^o$$

and set the sequence of taxes $\left\{\tau_{t+1}^k, \tau_t^c, \tau_{t+1}^\ell\right\}$ to satisfy,

$$\frac{B_{t+1}^o}{B_t^o} = \beta \frac{R_{t+1}}{\left(1 + \tau_{t+1}^k\right)} \frac{1}{\left(1 + \tau_{t+1}^c\right) / \left(1 + \tau_t^c\right)},$$

given the equilibrium rate R_{t+1} induced by his solution. This equilibrium has to be found to provide an actual implementation. Recall from the previous step that once we determine the saver's consumption path, we have the worker's expenditures. Hence, we are free to treat C^w, S^w, $\mu.$ and C^w, as functions of B in the Primal and Ramsey problems (assuming that both problems produce the same path of B_t^o) as we do in the problem without taxes in the main text:

$$S^w\left(B, \tilde{B}\right) \equiv \min\left\{\max\left\{\tilde{B} - B, 0\right\}, 1 - (1 - \beta) B\right\},$$

$$\mu\left(B, \tilde{B}\right) \equiv 1 - (1 - \beta) B - S^w\left(B, \tilde{B}\right),$$

$$X^w\left(B, \tilde{B}\right) \equiv \mathcal{A}\left(\mu\left(B, \tilde{B}\right)\right)\left(1 - (1 - \beta) B - S^w\left(B, \tilde{B}\right)\right),$$

and

$$C^w\left(B, \tilde{B}\right) = X^w\left(B, \tilde{B}\right) + S^w\left(B, \tilde{B}\right).$$

Recall that the Ramsey planner must satisfies the two household Euler equations:

$$\frac{C^s_{t+1}}{C^s_t} \frac{1}{\beta} \left[\frac{1 + \tau^c_{t+1}}{1 + \tau^c_t}\right] \left(1 + \tau^k_t\right) = R_{t+1} \tag{44}$$

and for the worker, at continuity points,

$$\frac{C^w_{t+1}}{C^w_t} \frac{1}{\beta} \left[\frac{1 + \tau^c_{t+1}}{1 + \tau^c_t}\right] \left[\frac{1 + \left(\mathcal{A}\left(\mu_{t+1}\right)^{-1} - 1\right) \mathbb{I}_{[S^w_{t+1}=0]}}{1 + \left(\mathcal{A}\left(\mu_t\right)^{-1} - 1\right) \mathbb{I}_{[X_t>0]}}\right] = R_{t+1}, \tag{45}$$

where I express the indicators as a function of consumption since these conditions are equivalent to the ones in the main text and, likewise, I work directly with consumption. I verify below, in the solution to the Primal Planner's Problem, that the planner never chooses $B^o_t = \tilde{B}_t$ but may chose $B^o_t = B^\star_t$. Thus, the worker's Euler equation must satisfy:

$$\lim_{B^o_t \uparrow B^\star_t} \frac{C^w_{t+1}}{C^w_t} \frac{1}{\beta} \left[\frac{1 + \tau^c_{t+1}}{1 + \tau^c_t}\right] \left[\frac{1 + \left(\mathcal{A}\left(\mu_{t+1}\right)^{-1} - 1\right) \mathbb{I}_{[S^w_{t+1}=0]}}{1 + \left(\mathcal{A}\left(\mu_t\right)^{-1} - 1\right) \mathbb{I}_{[X_t>0]}}\right] < R_{t+1},$$

and

$$\lim_{B^o_t \downarrow B^\star_t} \frac{C^w_{t+1}}{C^w_t} \frac{1}{\beta} \left[\frac{1 + \tau^c_{t+1}}{1 + \tau^c_t}\right] \left[\frac{1 + \left(\mathcal{A}\left(\mu_{t+1}\right)^{-1} - 1\right) \mathbb{I}_{[S^w_{t+1}=0]}}{1 + \left(\mathcal{A}\left(\mu_t\right)^{-1} - 1\right) \mathbb{I}_{[X_t>0]}}\right] \geq R_{t+1}.$$

The numerator is the same in both cases, it equals 1, in the neighborhood of $B^\star_t$. If the worker's Euler equation holds with equality in the limit from above $B^\star_t$, we immediately verify that the inequality holds in the limit from below:

$$\lim_{B^o_t \downarrow B^\star_t} 1 + \left(\mathcal{A}\left(\mu_t\right)^{-1} - 1\right) \mathbb{I}_{[X_t>0]} = \delta^{-1} < 0 = \lim_{B^o_t \uparrow B^\star_t} 1 + \left(\mathcal{A}\left(\mu_t\right)^{-1} - 1\right) \mathbb{I}_{[X_t>0]}.$$

Hence, we are free to substitute the the strict inequality in the denominator for an equality:

$$\frac{C^w_{t+1}}{C^w_t} \frac{1}{\beta} \left[\frac{1 + \tau^c_{t+1}}{1 + \tau^c_t}\right] \left[\frac{1 + \left(\mathcal{A}\left(\mu_{t+1}\right)^{-1} - 1\right) \mathbb{I}_{[S^w_{t+1}=0]}}{1 + \left(\mathcal{A}\left(\mu_t\right)^{-1} - 1\right) \mathbb{I}_{[X_t\geq0]}}\right] = R_{t+1}.$$

I use this modified Euler equation in the rest of the proof.

Substituting out $\left[\frac{1+\tau^c_t}{1+\tau^c_{t+1}}\right] \beta R_{t+1}$ from both (44) and (45), and replacing the saver's

optimal expenditures, we obtain:

$$\left(1+\tau_t^k\right) = \frac{C_t^s}{C_{t+1}^s} \frac{C_{t+1}^w\left(B_{t+1}, \tilde{B}_{t+1}\right)}{C_t^w\left(B_t, \tilde{B}_t\right)} \left[\frac{1 + \left(\mathcal{A}\left(\mu\left(B_{t+1}, \tilde{B}_{t+1}\right)\right)^{-1} - 1\right)\mathbb{I}_{[S^w\left(B_{t+1}, \tilde{B}_{t+1}\right)=0]}}{1 + \left(\mathcal{A}\left(\mu\left(B_t, \tilde{B}_t\right)\right)^{-1} - 1\right)\mathbb{I}_{[X_t \geq 0]}}\right].$$

We can treat the solution of this equation as a function mapping the sequence of solutions in the Primal Planner to the Ramsey planner:

$$\tau^k\left(B, B', \tilde{B}, \tilde{B}'\right) = \frac{B}{B'}\frac{1 - (1-\beta)B}{1 - (1-\beta)B'}\frac{Q\left(B', \tilde{B}'\right)}{Q\left(B, \tilde{B}\right)}\left[\frac{1 + \left(\mathcal{A}\left(\mu\left(B', \tilde{B}'\right)\right)^{-1} - 1\right)\mathbb{I}_{[S^w\left(B', \tilde{B}'\right)=0]}}{1 + \left(\mathcal{A}\left(\mu\left(B, \tilde{B}\right)\right)^{-1} - 1\right)\mathbb{I}_{[X_t \geq 0]}}\right] - 1.$$

(46)

As long as we have the sequence of debt obtained from the Primal Problem, we obtain a mapping from this solution to the sequence of capital taxes.

The equilibrium rate is deduced from (44),

$$R_{t+1} = \left[\frac{1+\tau_{t+1}^c}{1+\tau_t^c}\right]\frac{B_{t+1}^o}{B_t^o}\frac{1}{\beta}\left(1 + \tau^k\left(B_t^o, B_{t+1}^o, \tilde{B}_t, \tilde{B}_{t+1}\right)\right).$$

Hence, other than for time zero, expenditure taxes are indeterminate. Indeed, any sequence of expenditure taxes satisfies the saver's budget equation. If we substitute (44) into the saver budget constraint to obtain:

$$\frac{C_t^s}{C_{t+1}^s}\beta\left[\frac{1+\tau_t^c}{1+\tau_{t+1}^c}\right]B_{t+1} + (1+\tau_t^c)C_t^s = B_t.$$

Using that $C_t^s = (1-\beta)B_t^o$ and that the debt induced by the Ramsey solution must satisfy $B_t = B_t^o/(1+\tau_t^c)$, the budget constraint is verified.

Since there are multiple paths for expenditure taxes and labor income taxes, the only condition need is that they jointly satisfy the government budget constraint. Replacing the results above, we obtain that any sequence $\left\{\tau_t^c, \tau_{t+1}^\ell\right\}$ that satisfies:

$$\frac{\tau^k\left(B_t^o, B_{t+1}^o, \tilde{B}_t, \tilde{B}_{t+1}\right)}{1 + \tau^k\left(B_t^o, B_{t+1}^o, \tilde{B}_t, \tilde{B}_{t+1}\right)}\left[\frac{1+\tau_t^c}{1+\tau_{t+1}^c}\right]\beta B_t^o + \tau_t^c\left((1-\beta)B_t^o + \frac{1 - (1-\beta)B_t^o}{Q\left(B_t^o, \tilde{B}_t\right)}\right) + \tau_{t+1}^\ell = 0,$$

implements the Primal Planner allocation. For an implementation where $\tau_t^c = 0$, $\forall t \geq 1$, Hence, we have:

$$\tau_t^\ell = -\frac{\tau^k \left(B_t^o, B_{t+1}^o, \tilde{B}_t, \tilde{B}_{t+1} \right)}{1 + \tau^k \left(B_t^o, B_{t+1}^o, \tilde{B}_t, \tilde{B}_{t+1} \right)} \beta B_t^o,$$

also implements the solution.

C.1.1 Auxiliary Proofs

Proof of Lemma 13. To proof the result, I solve the saver's problem for an arbitrary sequence of taxes:

Problem 9. *The saver's problem with taxes is:*

$$V_t = \sum_{t \geq 0} \beta^t \log \left(C_t \right)$$

subject to: $\left(1 + \tau_{t+1}^k \right) R_{t+1}^{-1} B_{t+1} + \left(1 + \tau_t^c \right) C_t = B_t$, *with B_0 given.*

Dividing both sides of the budget constraint by $1 + \tau_t^c$ and multiplying and dividing by $\left(1 + \tau_{t+1}^c \right)$ in the first term, we obtain:

$$\left(1 + \tau_{t+1}^k \right) \frac{B_{t+1}}{R_{t+1} \left(1 + \tau_t^c \right)} \frac{\left(1 + \tau_{t+1}^c \right)}{\left(1 + \tau_{t+1}^c \right)} + C_t = \frac{B_t}{\left(1 + \tau_t^c \right)}.$$

I introduce the following change of variable:

$$B_t^o \equiv \frac{B_t}{1 + \tau_t^c}.$$

Using this change of variables, the budget constraint is modified to:

$$B_{t+1}^o = \hat{R}_{t+k} \left(B_t^o - C_t \right),$$

where

$$\hat{R}_{t+k} \equiv \frac{R_{t+1}}{\left(1 + \tau_{t+1}^k \right)} \frac{1}{\left(1 + \tau_{t+1}^c \right) / \left(1 + \tau_t^c \right)}.$$

This change of variables implies that the original problem can be reformulated as follows.

Problem 10. *Equivalent problem*

$$V_t = \sum_{t \geq 0} \beta^t \log(C_t)$$

subject to: $\hat{R}_{t+k}^{-1} B_{t+1}^o + C_t = B_t^o$, *where* $B_0^o \equiv (1 + \tau_0^c)^{-1} B_0$.

The solution to this problem is typical of log. Conjecture that:

$$V_t = V(B^o, t) = \frac{1}{1 - \beta} \log(D) + v(t).$$

We thus have that:

$$V(B^o, t) = \max_C \ \log(C) + \beta \log\left(\hat{R}_{t+1}(B^o - C)\right) + \beta v(t+1).$$

$$= \max_C \ \log(C) + \frac{\beta}{1-\beta} \log(B^o - C) + \beta\left(\frac{1}{1-\beta} \log\left(\hat{R}_{t+1}\right) + v(t+1)\right).$$

Taking first-order conditions with respect to C, we obtain:

$$\frac{1}{C} = \frac{1}{B^o - C} \frac{\beta}{1 - \beta} \rightarrow C = (1 - \beta) B^o.$$

We verify the conjecture by replacing the expenditure rule:

$$V(B^o, t) = \frac{\log(B^o)}{(1 - \beta)} + \frac{\log(1 - \beta)}{1 - \beta} + \beta\left(\frac{1}{1 - \beta} \log\left(\hat{R}_{t+1}\right) + v(t+1)\right),$$

where

$$v(t) = \frac{\log(1 - \beta) + \beta \log\left(\hat{R}_{t+1}\right)}{1 - \beta} + \beta v(t+1).$$

C.2 Proof of Proposition 8

I first state Proposition 8 in greater generality than as shown in the main body of the paper.

Proposition 14. (Solution of the Primal Problem): *The solution to the Primal Planner Problem is given by the solution to the following static problem:*

Problem 11.

$$\mathcal{P}^\theta\left(\tilde{B}\right) = \max_{B \in \left[0,\bar{B}\right]} \mathcal{P}\left(B, \tilde{B}\right)$$

where

$$\mathcal{P}\left(B, \tilde{B}\right) \equiv \left\{(1-\theta)\log\left((1-\beta)B\right) + \theta\log\left(\mathcal{A}\left(\mu\left(B,\tilde{B}\right)\right)\mu\left(B,\tilde{B}\right) + S^w\left(B,\tilde{B}\right)\right)\right\}$$

and $\mu\left(B, \tilde{B}\right)$ *and* $S^w\left(B, \tilde{B}\right)$:

$$\mu\left(B,\tilde{B}\right) \equiv 1 - (1-\beta)B - \min\left\{\max\left\{\tilde{B} - B, 0\right\}, 1 - (1-\beta)B\right\}$$

$$S^w\left(B,\tilde{B}\right) \equiv \min\left\{\max\left\{\tilde{B} - B, 0\right\}, 1 - (1-\beta)B\right\}.$$

Let the solution to this problem be $\mathcal{B}^\theta$. Then, the solution to the primal planner's problem is $B_t = \mathcal{B}^\theta\left(\tilde{B}_t\right)$. The function $\mathcal{B}^\theta$ satisfies:

I. Efficiency Threshold. For $\tilde{B} \geq \frac{1-\theta\beta}{1-\beta}$, $\mathcal{B}^\theta = B_{ss}$. Moreover, for this debt level $X^w = 0$.

II. Inefficiency Threshold. For $\tilde{B} < \frac{1-\theta\beta}{1-\beta}$ the planner's solution induces TFP losses. The solution to the Primal Planner's problem in this region depends on the threshold SBL, $\tilde{B}^e$.

II.a Social Insurance and Productive Efficiency. For $\tilde{B} \in \left[\tilde{B}^i, \frac{1-\theta\beta}{1-\beta}\right]$, we have that $\mathcal{B}^\theta = B^\star\left(\tilde{B}\right)$:

$$\frac{1 - (1-\beta)B^\star\left(\tilde{B}\right)}{(1-\beta)B^\star\left(\tilde{B}\right)} \leq \frac{\theta}{1-\theta}\frac{1-\beta\delta}{1-\beta}$$

II.b Social Insurance and Complements Productive Efficiency. *If $\tilde{B}^s < \tilde{B}^i$, for $\tilde{B} \in \left[\tilde{B}^s, \tilde{B}^i\right]$, we have that $\mathcal{B}^\theta = B^\star\left(\tilde{B}\right)$ if:*

$$\frac{1 - (1-\beta) B^\theta}{(1-\beta) B^\theta} = \frac{\theta}{1-\theta} \frac{Q\left(B^\theta, \tilde{B}\right)}{q\left(B^\theta, \tilde{B}\right)} \left(\frac{q\left(B^\theta, \tilde{B}\right) - \beta\left(1 + \epsilon_\mu^{\mathcal{A}}\left(\mu\left(B^\theta, \tilde{B}\right)\right)\right)}{1-\beta}\right).$$

Moreover, for this debt level $X^w, S^w > 0$.

II.c Social Insurance conflicts Productive Efficiency. *For $\tilde{B} \in \left[0, \tilde{B}^s\right]$, we have that $\mathcal{B}^\theta$ is the unique constant solution $B^\theta > B_{ss}$ to the equation:*

$$\frac{1 - (1-\beta) B^\theta}{(1-\beta) B^\theta} = \frac{\theta}{1-\theta} \left(1 + \epsilon_\mu^{\mathcal{A}}\left(1 - (1-\beta) B^\theta\right)\right).$$

Moreover, for this debt level $S^w = 0$.

Threshold value $\tilde{B}^i$.

$\tilde{B}^i$ is the unique solution to:

$$\frac{1 - (1-\beta) B^\star\left(\tilde{B}^i\right)}{(1-\beta) B^\star\left(\tilde{B}^i\right)} = \frac{\theta}{1-\theta} \frac{1 - \beta\delta}{1-\beta}$$

Threshold value $\tilde{B}^s$. *Let $\underline{\mathcal{P}}^\theta \equiv \mathcal{P}^\theta(0)$ and $\bar{\mathcal{P}}^\theta \equiv \mathcal{P}^\theta\left(\bar{B}\right)$. The threshold $\tilde{B}^s$ solves:*

$$\mathcal{P}^\theta = \underline{\mathcal{P}}^\theta + \int_{\tilde{B}^s}^{\frac{1-\theta\beta}{1-\beta}} \left(\mathcal{P}_{\tilde{B}} + \mathcal{P}_B|_{B=B^\star(\tilde{B})} \cdot B_{\tilde{B}}^\star\left(\tilde{B}\right)\right) d\tilde{B}.$$

where for $\tilde{B} \in \left[\tilde{B}^e, \frac{1-\theta\beta}{1-\beta}\right]$ we have that $\mathcal{P}_{\tilde{B}}^\theta\left(\tilde{B}\right)$ equals:

$$= \frac{\theta}{1 - (1-\beta) B^p(\tilde{B})} \frac{Q\left(B^p(\tilde{B}), \tilde{B}\right)}{q\left(B^p(\tilde{B}), \tilde{B}\right)} \log\left(\mathcal{A}\left(\mu\left(B^p(\tilde{B}), \tilde{B}\right)\right) - \left(1 + \epsilon_\mu^{\mathcal{A}}\left(\mu\left(B^p(\tilde{B}), \tilde{B}\right)\right)\right)\right)$$

and

$$\mathcal{P}_B|_{B=B^\star(\tilde{B})} = \frac{1-\theta}{(1-\beta)\,B^\star\left(\tilde{B}\right)} - \frac{\theta\,(1-\beta\delta)}{1-(1-\beta)\,B^\star\left(\tilde{B}\right)}$$

To begin the proof, let me start with the Primal Problem in the statement of Proposition 7. Taking the sequence of borrowing limits $\left\{\tilde{B}_t\right\}_{t\geq 0}$, the Primal Planner maximizes:

$$\max_{\{B_t\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \left[(1-\theta)\log\left(C_t^s\right) + \theta\log\left(X_t^w + S_t^w\right)\right],$$

subject to the saver's budget constraint,

$$C_t^s = (1-\beta)\,B_t,\ \forall t \geq 0,$$

the income expenditure identity,

$$1 = \mu_t + S_t^w + (1-\beta)\,B_t,$$

the spot expenditure constraint,

$$S_t \leq \max\left\{\tilde{B}_t - B_t, 0\right\},\ \ \forall t \geq 0,$$

and the cost of chained goods,

$$X_t^w = \mathcal{A}\left(\mu_t\right)\mu_t,$$

and subject to $\mu_t \in [0,1]$.

We have that for any B_t, the consumption delivered to the savers is fixed. To maximize the worker's utility, we must

$$\max\log\left(X_t^w + S_t^w\right)$$

subject to:

$$1 - (1-\beta)\,B_t = \mu_t + S_t^w$$

$$S_t \leq \max\left\{\tilde{B}_t - B_t, 0\right\}$$

$$X_t^w = \mathcal{A}\left(\mu_t\right)\mu_t.$$

and subject to $\mu_t \in [0, 1]$.

The last constraint implies:

$$S_t^w \leq 1 - (1 - \beta) B_t.$$

Hence, the Primal Planner respects the same constraint as the worker:

$$S_t^w = \min \left\{ \max \left\{ \tilde{B}_t - B_t, 0 \right\}, 1 - (1 - \beta) B_t \right\}.$$

Therefore, the expenditures on chained goods are:

$$\mu_t = 1 - (1 - \beta) B_t - \min \left\{ \max \left\{ \tilde{B}_t - B_t, 0 \right\}, 1 - (1 - \beta) B_t \right\}.$$

Since the the maximization is static, we can solve it state by state as in the statement of the proposition. Thus, the objective of the Primal Planner is the same as solving the following problem at each date:

Problem 12. *The Primal Planner's problem is given by:*

$$\mathcal{P}^\theta \left(\tilde{B} \right) = \max_{B \in [0, \bar{B}]} \mathcal{P} \left(B, \tilde{B} \right)$$

where

$$\mathcal{P} \left(B, \tilde{B} \right) = \left\{ (1 - \theta) \log \left((1 - \beta) B \right) + \theta \log \left(\mathcal{A} \left(\mu \left(B, \tilde{B} \right) \right) \mu \left(B, \tilde{B} \right) + S^w \left(B, \tilde{B} \right) \right) \right\}$$

subject to:

$$\mu \left(B, \tilde{B} \right) \equiv 1 - (1 - \beta) B - \min \left\{ \max \left\{ \tilde{B} - B, 0 \right\}, 1 - (1 - \beta) B \right\}$$

$$S^w \left(B, \tilde{B} \right) = \min \left\{ \max \left\{ \tilde{B} - B, 0 \right\}, 1 - (1 - \beta) B \right\}.$$

This is a problem where the planner can distribute wealth at will, but respects the constraints. Next, I proceed to solve this problem. Naturally, welfare depends on whether and how the planner may want to distort TFP to provide insurance. There are multiple policy regimes that depend on the SBL, $\tilde{B}$. I make use of two problems, the best and

worst value problems:

$$\underline{\mathcal{P}}^{\theta} \equiv \mathcal{P}^{\theta}\left(0\right) \text{ and } \bar{\mathcal{P}}^{\theta} \equiv \mathcal{P}^{\theta}\left(\bar{B}\right).$$

Case 1. Values of $\tilde{B}$ such that all consumption is spot.   Ideally, the planner wants to maximize spot consumption and set $\mu = 0$. The unconstrained solution to the Primal Planner's problem is given by the ratio of of Pareto weights:

$$\frac{1 - (1 - \beta) B^o}{(1 - \beta) B^o} = \frac{\theta}{1 - \theta} \to B^o = \frac{1 - \theta}{1 - \beta} = B_{ss}.$$

This yields the same value as $\bar{\mathcal{P}}^{\theta}$.

This level of debt must satisfy the condition that all spot consumption must be feasible:

$$\max\left\{\tilde{B} - B^o, 0\right\} \geq 1 - (1 - \beta) B^o > 0.$$

Thus, we need

$$\tilde{B} \geq B^o$$

and that:

$$\tilde{B} \geq 1 + \beta B^o.$$

Combining both constraints we have:

$$\tilde{B} \geq B^o + \max\left\{1 - (1 - \beta) B^o, 0\right\}.$$

We know that the optimal debt B^o must be less than the natural borrowing limit. Hence, the inequality is just:

$$\tilde{B} \geq 1 + \beta B^o = 1 + \frac{\beta}{1 - \beta}(1 - \theta) = \frac{1 - \theta\beta}{1 - \beta}. \tag{47}$$

Thus, for these levels of the SBL, the planner can achieve the unconstrained solution. This corresponds to the debt in the efficient steady state level of the competitive equilibrium that produces the planner's Pareto weights.

Case 2. Values of $\tilde{B}$ such that some consumption is chained. Now consider the case where the constraint binds, $\tilde{B} < \frac{1-\theta\beta}{1-\beta}$. In this case, the planner cannot achieve the unconstrained solution. The amount of chained expenditures are therefore positive:

$$\mu\left(B, \tilde{B}\right) = 1 - (1-\beta)B - \min\left\{\max\left\{\tilde{B} - B, 0\right\}, 1 - (1-\beta)B\right\} > 0.$$

We have critical values.

- If the planner chooses $B^p \geq B_h\left(\tilde{B}\right) = \tilde{B}$, there is no spot consumption.

- If the planner chooses $B^p < B_l\left(\tilde{B}\right) < \tilde{B}$, that there is no chained consumption. This threshold $B_l\left(\tilde{B}\right)$ solves,

$$\max\left\{\tilde{B} - B_l, 0\right\} = 1 - (1-\beta)B_l$$

  when since $B_l < \tilde{B}$, we obtain:

$$\tilde{B} = 1 + \beta B_l \rightarrow B_l\left(\tilde{B}\right) = \max\left\{0, \beta^{-1}\left(\tilde{B} - 1\right)\right\} = B^\star\left(\tilde{B}\right).$$

  Obviously, $B^\star\left(\tilde{B}\right) < B^o$ since we are in the constrained region.

Consider now the planner problem that restricts choices to at least some of both goods are consumed by the worker:

Problem 13. *The Primal Planner's problem restricted to both types of consumption is:*

$$\tilde{\mathcal{P}}\left(\tilde{B}\right) = \max_B\left\{(1-\theta)\log\left((1-\beta)B\right) + \theta\log\left(\mathcal{A}\left(\mu\left(B, \tilde{B}\right)\right) \cdot \mu\left(B, \tilde{B}\right) + S^w\left(B, \tilde{B}\right)\right)\right\}$$

subject to:

$$B \in \left[B_l\left(\tilde{B}\right), B_h\left(\tilde{B}\right)\right]$$

$$\mu\left(B, \tilde{B}\right) \equiv 1 + \beta B - \tilde{B}$$

and

$$S^w\left(B, \tilde{B}\right) = \tilde{B} - B.$$

We have the following Lemma.

Lemma 14. *For any $\tilde{B} < \frac{1-\theta\beta}{1-\beta}$, the original planner problem satisfies:*

$$\mathcal{P}^\theta\left(\tilde{B}\right) = \max\left\{\tilde{\mathcal{P}}\left(\tilde{B}\right), \underline{\mathcal{P}}^\theta\right\}.$$

Proof. Indeed, in the region $B^p \in \left[0, B_l\left(\tilde{B}\right)\right]$ the objective of the planner is equivalent to the objective when the SBL is most relaxed, $\mathcal{P}\left(B, \bar{B}\right)$. Thus, since $\tilde{B} < \frac{1+\theta\beta}{1-\beta}$, the planner's objective is increasing in the region $\left[0, B_l\left(\tilde{B}\right)\right]$. Thus, the planner's solution must fall in between $B^p \in \left[B_l\left(\tilde{B}\right), \bar{B}\right]$. For any $B^p \geq B_h\left(\tilde{B}\right) = \tilde{B}$, the objective function in $\mathcal{P}\left(B, \tilde{B}\right)$ is independent of $\tilde{B}$ and hence, must coincide with the value of $\underline{\mathcal{P}}^\theta$. Hence, we can partition $\mathcal{P}^\theta\left(\tilde{B}\right)$ according to the Lemma.

$\square$

To prove the main result, I solve the problems $\underline{\mathcal{P}}^\theta$ and $\tilde{\mathcal{P}}\left(\tilde{B}\right)$

Auxiliary Problem $\underline{\mathcal{P}}^\theta$: no spot consumption. The planner's problem with the tightest SBL $\underline{\mathcal{P}}^\theta = \mathcal{P}^\theta(0)$ is given by:

$$\underline{\mathcal{P}}^\theta = \max_{B\in\left[0,\bar{B}\right]} \mathcal{P}(B, 0)$$

where

$$\underline{\mathcal{P}}^\theta = \max_{B\in\left[0,\bar{B}\right]} \left\{(1 - \theta)\log\left((1 - \beta)B\right) + \theta\log\left(\mathcal{A}\left(\mu\left(B, 0\right)\right)\mu\left(B, 0\right)\right)\right\}$$

subject to:

$$\mu\left(B, 0\right) \equiv 1 - (1 - \beta)B.$$

To solve this problem, I perform some calculations. First, note that:

$$\frac{\partial\left[\mathcal{A}\left(\mu\right)\mu\right]}{\partial\mu} = \mathcal{A}\left(\mu\right)\left(1 + \epsilon^\mathcal{A}\right)$$

where,

$$\epsilon_\mu^\mathcal{A} \equiv \frac{\partial\mathcal{A}\left(\mu\right)}{\partial\mu}\frac{\mu}{\mathcal{A}\left(\mu\right)}.$$

The derivative $\underline{\mathcal{P}}_B$ is therefore given by:

$$\underline{\mathcal{P}}_B = (1 - \theta) \frac{(1 - \beta)}{(1 - \beta) B} + \theta \frac{\mathcal{A}(\mu)\left(1 + \epsilon_\mu^{\mathcal{A}}\right)\mu_B(B, 0)}{\mathcal{A}(\mu)\mu} = \frac{(1 - \theta)}{B} - \theta \frac{\left(1 + \epsilon_\mu^{\mathcal{A}}\right)(1 - \beta)}{1 - (1 - \beta) B}.$$

The second equation uses that $\mu_B(B, 0) \equiv -(1 - \beta)$.

The first term in $\underline{\mathcal{P}}_B$, $(1 - \theta)/B$, is decreasing in B. The second term,

$$\theta \frac{\left(1 + \epsilon_\mu^{\mathcal{A}}\right)(1 - \beta)}{1 - (1 - \beta) B}, \tag{48}$$

is increasing. We know this because denominator is decreasing in $B \in \left[0, \bar{B}\right]$ and the elasticity of TFP is itself decreasing in μ,

$$\epsilon_{\mu\mu}^{\mathcal{A}} = \frac{\partial}{\partial \mu}\left[\frac{\mathcal{A}'(\mu)\mu}{\mathcal{A}(\mu)}\right] = \frac{\mathcal{A}''(\mu)\mu}{\mathcal{A}(\mu)} + \frac{\mathcal{A}'(\mu)}{\mathcal{A}(\mu)} - \frac{[\mathcal{A}'(\mu)]^2}{[\mathcal{A}(\mu)]^2}\mu < 0.$$

Hence, $\epsilon_{\mu\mu}^{\mathcal{A}}\mu_B(B, 0) > 0$, since the product of two numbers thus, the numerator of the second term (48). Thus, $\underline{\mathcal{P}}$ is concave and therefore $\underline{\mathcal{P}}^\theta$ has a unique solution:

$$\frac{1 - (1 - \beta) B}{(1 - \beta) B} = \frac{\theta}{1 - \theta}\left(1 + \epsilon_\mu^{\mathcal{A}}(\mu(B, 0))\right)$$

and recall that $\mu(B, 0) = 1 - (1 - \beta) B$. I call this solution $\underline{B}^p$: the planner debt level under the most tight SBL. We have the following Lemma:

Lemma 15. *The solution $\underline{B}^p > B^o$.*

Proof. The proof is immediate from $1 + \epsilon_\mu^{\mathcal{A}} < 1$ and the fact that

$$\frac{1 - (1 - \beta) B^o}{(1 - \beta) B^o} = \frac{\theta}{(1 - \theta)}.$$

$\square$

Next, I solve $\tilde{\mathcal{P}}\left(\tilde{B}\right)$.

Auxiliary Problem $\tilde{\mathcal{P}}\left(\tilde{B}\right)$**: spot and chained consumption.**   Consider now the planner problem where at least some of both goods are consumed by the worker:

$$\tilde{\mathcal{P}}\left(\tilde{B}\right) = \max_{B \in \left[B^\star\left(\tilde{B}\right), \tilde{B}\right]} \tilde{\mathcal{P}}\left(B, \tilde{B}\right)$$

where

$$\tilde{\mathcal{P}}\left(\tilde{B}\right) \equiv \left\{(1 - \theta) \log\left((1 - \beta) B\right) + \theta \log\left(\mathcal{A}\left(\mu\left(B, \tilde{B}\right)\right) \cdot \mu\left(B, \tilde{B}\right) + S^w\left(B, \tilde{B}\right)\right)\right\},$$

subject to:

$$\mu\left(B, \tilde{B}\right) \equiv 1 + \beta B - \tilde{B}$$

and

$$S^w\left(B, \tilde{B}\right) = \tilde{B} - B.$$

The derivative of the objective in $\tilde{\mathcal{P}}\left(\tilde{B}\right)$ is:

$$\tilde{\mathcal{P}}_B\left(B, \tilde{B}\right) = (1 - \theta) \frac{1}{B} + \theta \frac{\mathcal{A}\left(\mu\left(B, \tilde{B}\right)\right)\left(1 + \epsilon_\mu^{\mathcal{A}}\left(\mu\left(B, \tilde{B}\right)\right)\right)\beta - 1}{\mathcal{A}\left(\mu\left(B, \tilde{B}\right)\right)\mu\left(B, \tilde{B}\right) + \tilde{B} - B}.$$

Recall that,

$$C^w\left(B, \tilde{B}\right) = \mathcal{A}\left(\mu\left(B, \tilde{B}\right)\right)\mu\left(B, \tilde{B}\right) + \tilde{B} - B.$$

and

$$Q \cdot C^w\left(B, \tilde{B}\right) = E^w\left(B, \tilde{B}\right) = 1 - (1 - \beta) B$$

Hence, using the definition of Q and q we rewrite:

$$\tilde{\mathcal{P}}_B\left(B, \tilde{B}\right) = (1 - \theta) \frac{1}{B} - \theta Q \frac{1 - \beta\left(1 + \epsilon_\mu^{\mathcal{A}}\left(\mu\left(B, \tilde{B}\right)\right)\right)\mathcal{A}\left(\mu\left(B, \tilde{B}\right)\right)}{1 - (1 - \beta) B}$$

We can multiply both sides by the ratio of $1 - (1 - \beta) B$ and divide by $(1 - \beta)(1 - \theta)$ and

obtain:

$$\tilde{\mathcal{P}}_B\left(B,\tilde{B}\right)\frac{1-(1-\beta)B}{(1-\theta)(1-\beta)B} = \frac{1-(1-\beta)B}{(1-\beta)B} - \frac{\theta}{1-\theta}Q\left(\frac{1-\beta\left(1+\epsilon_\mu^{\mathcal{A}}\right)\mathcal{A}(\mu)}{1-\beta}\right).$$

This function must have the same sign as $\tilde{\mathcal{P}}_B\left(B,\tilde{B}\right)$, since at was obtained by multiplication of positive numbers. The first term is decreasing in B. In turn, $Q_\mu\mu_B$ is increasing in B. Hence, as long as

$$\mathcal{A}(\mu)\left(1+\epsilon_\mu^{\mathcal{A}}\right) = \mathcal{A}(\mu) + \mathcal{A}'(\mu)\mu$$

is decreasing in B, the second term is increasing. The second term is indeed decreasing in μ since its derivative is:

$$2\mathcal{A}'(\mu) + \mathcal{A}''(\mu)\mu < 0,$$

where the sign follows immediately from the concavity and monotone decreasing properties of $\mathcal{A}$. Hence, the objective function $\tilde{\mathcal{P}}\left(B,\tilde{B}\right)$ is concave in B. Furthermore, since

$$Q\left(\frac{1-\beta\left(1+\epsilon_\mu^{\mathcal{A}}\right)\mathcal{A}(\mu)}{1-\beta}\right) > 1,$$

and we have an interior maximum in the region $B \in \left[B^\star\left(\tilde{B}\right), \tilde{B}\right]$, B must solve:

$$\frac{1-(1-\beta)B}{(1-\beta)B} = \frac{\theta}{1-\theta}Q\left(\frac{1-\beta\left(1+\epsilon_\mu^{\mathcal{A}}\right)\mathcal{A}(\mu)}{1-\beta}\right),$$

and must be such that $B < B^o$.

Next, we establish properties regarding the limits of this function at $B^\star\left(\tilde{B}\right)$ and $\tilde{B}$. I start with $B^\star\left(\tilde{B}\right)$:

$$\lim_{B\downarrow B^\star(\tilde{B})}\frac{1-(1-\beta)B}{(1-\beta)B} - \frac{\theta}{1-\theta}Q\left(\frac{1-\beta\left(1+\epsilon_\mu^{\mathcal{A}}\right)\mathcal{A}(\mu)}{1-\beta}\right) = \ldots$$

$$\frac{1-(1-\beta)\,B^\star\left(\tilde{B}\right)}{(1-\beta)\,B^\star\left(\tilde{B}\right)} - \frac{\theta}{1-\theta}\left(\frac{1}{1-\beta} - \frac{\beta\left(1+\lim_{\mu\downarrow0}\epsilon_\mu^{\mathcal{A}}\right)\lim_{\mu\downarrow0}\mathcal{A}\left(\mu\right)}{1-\beta}\right),$$

where I used $\lim_{\mu\downarrow0} Q = 1$. Appendix A, shows that $\lim_{\mu\downarrow0} 1+\epsilon_\mu^{\mathcal{A}} = 0$ and that $\lim_{\mu\downarrow0} \mathcal{A}\left(\mu\right) = \delta$. Thus, the limit of the objective function at the left boundary has the sign of:

$$\frac{1-(1-\beta)\,B^\star\left(\tilde{B}\right)}{(1-\beta)\,B^\star\left(\tilde{B}\right)} - \frac{\theta}{1-\theta}\frac{1-\beta\delta}{1-\beta} = \frac{1-(1-\beta)\,B^\star\left(\tilde{B}\right)}{(1-\beta)\,B^\star\left(\tilde{B}\right)} - \frac{1-(1-\beta)\,B^o}{(1-\beta)\,B^o}\frac{1-\beta\delta}{1-\beta}.$$

The function $B^\star\left(\tilde{B}\right) < B^o$ but $\frac{1-\beta\delta}{1-\beta} > 1$, hence the sign is ambiguous. The solution is at this corner if the function is negative.

Next, we consider the limit of the derivative of the objective at $\tilde{B}$. Evaluating the limits is immediate. Hence, if

$$\frac{1-(1-\beta)\,\tilde{B}}{(1-\beta)\,\tilde{B}} \geq \frac{\theta}{1-\theta}\left(\frac{1-\beta\left(1+\epsilon_\mu^{\mathcal{A}}\left(\mu\right)\right)\mathcal{A}\left(\mu\right)}{1-\beta}\right)\Bigg|_{\mu=\left(1-(1-\beta)\tilde{B}\right)}$$

the solution is $B = \tilde{B}$. Moreover, we know that since

$$\left(\frac{1-\beta\left(1+\epsilon_\mu^{\mathcal{A}}\right)\mathcal{A}\left(\mu\right)}{1-\beta}\right) > 1,$$

the corner solution $\tilde{B}$ is chosen only if

$$\frac{1-(1-\beta)\,\tilde{B}}{(1-\beta)\,\tilde{B}} > \frac{1-(1-\beta)\,B^o}{(1-\beta)\,B^o}.$$

Collecting the results up to this point, we have the following Lemma.

Lemma 16. *The solution* B_ℓ^p *to* $\tilde{\mathcal{P}}\left(\tilde{B}\right)$ *is as follows.*

I. $B_\ell^p = \tilde{B}$ *if*

$$\frac{1-(1-\beta)\,\tilde{B}}{(1-\beta)\,\tilde{B}} \geq \frac{\theta}{1-\theta}\left(\frac{1-\beta\left(1+\epsilon_\mu^{\mathcal{A}}\left(\mu\right)\right)\mathcal{A}\left(\mu\right)}{1-\beta}\right)\Bigg|_{\mu=\left(1-(1-\beta)\tilde{B}\right)},$$

II. $B_\ell^p = B^\star \left(\tilde{B} \right)$ *if*

$$\frac{1 - (1 - \beta) B^\star \left(\tilde{B} \right)}{(1 - \beta) B^\star \left(\tilde{B} \right)} \leq \frac{\theta}{1 - \theta} \frac{1 - \beta\delta}{1 - \beta}.$$

III. *Otherwise,* B_ℓ^p *solves:*

$$\frac{1 - (1 - \beta) B_\ell^p}{(1 - \beta) B_\ell^p} = \dots$$

$$\left. \frac{\theta}{1 - \theta} Q \left(B_\ell^p, \tilde{B} \right) \left(\frac{1 - \beta \left(1 + \epsilon_\mu^{\mathcal{A}} \left(\mu \left(B_\ell^p, \tilde{B} \right) \right) \right) \mathcal{A} \left(\mu \left(B_\ell^p, \tilde{B} \right) \right)}{1 - \beta} \right) \right|_{\mu = \left(1 - (1 - \beta) B_\ell^p \right)}.$$

The following Lemma characterizes threshold values for $\tilde{B}$ corresponding to the Lemma above:

Lemma 17. *The condition*

$$\frac{1 - (1 - \beta) B^\star \left(\tilde{B} \right)}{(1 - \beta) B^\star \left(\tilde{B} \right)} \leq \frac{\theta}{1 - \theta} \frac{1 - \beta\delta}{1 - \beta}$$

holds for all $\tilde{B} \geq \tilde{B}^i$ *such that:*

$$\frac{1 - (1 - \beta) B^\star \left(\tilde{B}^i \right)}{(1 - \beta) B^\star \left(\tilde{B}^i \right)} = \frac{\theta}{1 - \theta} \frac{1 - \beta\delta}{1 - \beta}.$$

The reverse inequality holds for $\tilde{B} \leq \tilde{B}^a$ *such that:*

$$\left. \frac{1 - (1 - \beta) B^\star \left(\tilde{B}^a \right)}{(1 - \beta) B^\star \left(\tilde{B}^a \right)} = \frac{\theta}{1 - \theta} \left(\frac{1 - \beta \left(1 + \epsilon_\mu^{\mathcal{A}} (\mu) \right) \mathcal{A} (\mu)}{1 - \beta} \right) \right|_{\mu = \left(1 - (1 - \beta) \tilde{B}^a \right)}.$$

Proof. The proof follows immediately from the fact that:

$$\frac{1 - (1 - \beta) B^\star \left(\tilde{B} \right)}{(1 - \beta) B^\star \left(\tilde{B} \right)},$$

is decreasing in $\tilde{B}$ and the function:

$$\frac{\theta}{1-\theta}Q\left(B,\tilde{B}\right)\left(\frac{1-\beta\left(1+\epsilon_\mu^\mathcal{A}\left(\mu\left(B,\tilde{B}\right)\right)\right)\mathcal{A}\left(\mu\left(B,\tilde{B}\right)\right)}{1-\beta}\right)\Bigg|_{\mu=(1-(1-\beta)B)}$$

increasing in B.

$\square$

Overall Solution. I showed above that:

$$\mathcal{P}^\theta\left(\tilde{B}\right)=\max\left\{\tilde{\mathcal{P}}\left(\tilde{B}\right),\underline{\mathcal{P}}^\theta\right\}.$$

The following Lemma shows that the solution B^p to the Planner's problem is never at $B^p=\tilde{B}$.

Lemma 18. *The planner never chooses $B^p=\tilde{B}$.*

Proof. To proof this Lemma observe that the left limit as $B\uparrow\tilde{B}$ satisfies

$$\frac{q-1-\epsilon^\mathcal{A}}{1-\beta}+1+\epsilon^\mathcal{A}>1+\epsilon^\mathcal{A},$$

where the inequality follows from $q>1$ and $\epsilon^\mathcal{A}<0$. As a consequence,

$$\frac{1-(1-\beta)\tilde{B}}{(1-\beta)\tilde{B}}\geq\frac{\theta}{1-\theta}\lim_{B\uparrow\tilde{B}}\Lambda\left(B,\tilde{B}\right)$$

implies

$$\frac{1-(1-\beta)\tilde{B}_t}{(1-\beta)\tilde{B}_t}>\frac{\theta}{1-\theta}\lim_{B\downarrow\tilde{B}_t}\Lambda\left(B,\tilde{B}\right).$$

Hence, although the derivative of the objective is discontinuous at $\tilde{B}$, we know that if the derivative is weakly increasing from the left and increasing from the right. This implies that $B=\tilde{B}$ is never an optimal choice.

$\square$

Next, observe $\tilde{\mathcal{P}}\left(\tilde{B}\right)$ has a compact-valued and continuous constraint correspondence with a continuous objective. Hence, it satisfies the conditions for the Theorem

of the Maximum. In addition it is immediate to verify that:

$$\lim_{\tilde{B}\uparrow\frac{1-\theta\beta}{1-\beta}} \tilde{\mathcal{P}}\left(\tilde{B}\right) = \bar{\mathcal{P}}^\theta \text{ and } \lim_{\tilde{B}\uparrow\frac{1-\theta\beta}{1-\beta}} B^p(\tilde{B}) = B^o.$$

I employ the Envelope Theorem on $\tilde{\mathcal{P}}\left(\tilde{B}\right)$. In the region where the solution to $\tilde{\mathcal{P}}\left(\tilde{B}\right)$ is not at a corner solution, the Envelope Theorem yields:

$$\tilde{\mathcal{P}}_{\tilde{B}} = \frac{\theta}{1-(1-\beta)\,B^p(\tilde{B})} Q\left(B^p(\tilde{B}),\tilde{B}\right)\left(1-\left(1+\epsilon_\mu^{\mathcal{A}}\left(\mu\left(B^p(\tilde{B}),\tilde{B}\right)\right)\right)\mathcal{A}\left(\mu\left(B^p(\tilde{B}),\tilde{B}\right)\right)\right) > 0,$$

since $\tilde{B}$ appears directly through μ and S in the objective.

The function is strictly increasing in $\tilde{B}$ since $\epsilon_\mu^{\mathcal{A}} < 0$ and $\mathcal{A} < 1$. In the region where the function is at the lower corner of the constraint:

$$B = B^\star\left(\tilde{B}\right),$$

the value of this term is:

$$\tilde{\mathcal{P}}_{\tilde{B}} = \frac{\theta}{1-(1-\beta)\,B^\star\left(\tilde{B}\right)}\,(1-\delta) > 0$$

and

$$\tilde{\mathcal{P}}_B\big|_{B=B^\star(\tilde{B})} = \frac{1-\theta}{(1-\beta)\,B^\star\left(\tilde{B}\right)} - \frac{\theta\,(1-\beta\delta)}{1-(1-\beta)\,B^\star\left(\tilde{B}\right)}.$$

Thus, we have that the marginal objective value is:

$$\tilde{\mathcal{P}}_{\tilde{B}} + \tilde{\mathcal{P}}_B\left(B^p(\tilde{B})\right) = \frac{1-\theta}{(1-\beta)\,B^\star\left(\tilde{B}\right)} + \frac{\beta\delta}{1-(1-\beta)\,B^\star\left(\tilde{B}\right)} > 0.$$

Hence, the envelope condition guarantees that $\tilde{\mathcal{P}}$ is decreasing in $\tilde{B}$. We have that: There exists a threshold $\tilde{B}^s$ such that:

$$\tilde{\mathcal{P}}\left(\tilde{B}^s\right) = \underline{\mathcal{P}}^\theta.$$

In particular, it solves:

$$\bar{\mathcal{P}}^{\theta} = \underline{\mathcal{P}}^{\theta} + \int_{\tilde{B}^s}^{\frac{1-\theta\beta}{1-\beta}} \left(\tilde{\mathcal{P}}_{\tilde{B}} + \tilde{\mathcal{P}}_B \left(B^p(\tilde{B}) \right) B^{\star}_{\tilde{B}} \left(\tilde{B} \right) \right) d\tilde{B}.$$

Notice that $\underline{\mathcal{P}}^{\theta}$ is finite and lower than the value of the problem without constraints, $\tilde{\mathcal{P}}(0)$ tends to $-\infty$ and $\tilde{\mathcal{P}}\left(\overline{B}\right)$ tends to the value of problem without constraints. This implies that $\tilde{\mathcal{P}}(0) < \underline{\mathcal{P}}^{\theta} < \tilde{\mathcal{P}}\left(\overline{B}\right)$ and from continuity of the value function (Theorem of the Maximum) and the Intermediate Value Theorem, the existence of $\tilde{B}^e$ is guaranteed. The solution to $\tilde{B}^e$ follows from the fundamental theorem of calculus.

Recall, that in the previous Lemma we had showed that $\tilde{B}$ is never a solution to $\tilde{\mathcal{P}}$ and the monotonicity the existence of the threshold B^i. These thresholds segments the intervals as given by the proposition. This concludes the proof of the general version of Proposition 8.

C.3 Proof of Proposition 9

Let's define the indirect social utility function of the Planner Problem with expenditures $\mathcal{P}^g$:

$$\mathcal{P}^g\left(B,\tilde{B},G^s,G^x\right) = (1-\theta)\log\left(C^s\left(B\right)\right) + \theta\log\left(S^w\left(B,\tilde{B},G^s,G^x\right) + X^w\left(B,\tilde{B},G^s,G^x\right)\right).$$

The following functions determine the allocation:

$$C^s\left(B\right) \equiv (1-\beta)B$$

$$E^w\left(B,G^s,G^x\right) \equiv 1 - E^s\left(B\right) - G^x - G^s$$

$$S^w\left(B,\tilde{B},G^s,G^x\right) \equiv \min\left\{\max\left\{\tilde{B}-B,0\right\},E^w\right\}$$

$$X^w\left(B,\tilde{B},G^s,G^x\right) \equiv \frac{E^w - \min\left\{\max\left\{\tilde{B}-B,0\right\},E^w\right\}}{q}$$

$$q\left(B,\tilde{B},G^s,G^x\right) \equiv \mathcal{A}^{-1}\left(\mu\right)$$

$$\mu\left(B,\tilde{B},G^s,G^x\right) \equiv G^x + X^w q.$$

The worker's total consumption is:

$$C^w = S^w\left(B,\tilde{B},G^s,G^x\right) + X^w\left(B,\tilde{B},G^s,G^x\right).$$

Because

$$\mathcal{Y} = C^w + C^S + G^s + \frac{G^x}{q\left(\mu\right)},$$

but $C^s\left(B\right)$ is independent of the government's expenditure, we obtain: $d\mathcal{Y} = dC^w + dG^s + d\left(G^x/q\left(\mu\right)\right)$.

Thus, we have that the government's expenditure multiplier for expenditure of type $i = x, s$ relates to the worker's consumption as follow:

$$\mathcal{M}^s\left(B,\tilde{B}\right) \equiv \frac{d\mathcal{Y}}{dG^s} = \frac{dC^w}{dG^s} + 1$$

and

$$\mathcal{M}^x \left(B, \tilde{B} \right) \equiv \frac{d\mathcal{Y}}{dG^x} = \frac{dC^w}{dG^x} + \frac{1}{q} - \frac{G^x}{q^2} \frac{dq}{d\mu} \frac{d\mu}{dG^x}$$
$$= \frac{dC^w}{dG^x} + \mathcal{A}\left(\mu\right).$$

where the last equation is because we are interested in obtaining the government's infinitesimal multiplier, evaluated at $G^x = G^s = 0$. We have defined these multipliers as:

$$\mathcal{M}^i \left(B, \tilde{B} \right) \equiv \left. \frac{d\mathcal{Y}}{dG^i} \right|_{G^x = G^s = 0},$$

and relating to the indirect social utility function, the change in the objective is:

$$\frac{\theta}{C^w} \frac{dC^w}{dG^i},$$

for expenditure i.

Case 1. All spot consumption $B < B^\star$.   If there is only spot consumption:

$$C^s \left(B \right) \equiv \left(1 - \beta \right) B$$
$$C^w \left(B, \tilde{B}, G^s, G^x \right) \equiv 1 - \left(1 - \beta \right) B - G^x - G^s$$
$$X^w \left(B, \tilde{B}, G^s, G^x \right) \equiv 0$$
$$q \left(B, \tilde{B}, G^s, G^x \right) \equiv \mathcal{A}^{-1} \left(\mu \right)$$
$$\mu \left(B, \tilde{B}, G^s, G^x \right) \equiv G^x.$$

Since worker consumption is independent of q, we have that:

$$\frac{dC^w}{dG^i} = -1$$

for both $i \in \{x, s\}$. In turn, we have that

$$\mathcal{M}^s\left(B, \tilde{B}\right) = \frac{dC^w}{dG^s} + 1 = 0.$$

Likewise, for chained expenditures we have:

$$\mathcal{M}^x\left(B, \tilde{B}\right) = \frac{dC^w}{dG^x} + \mathcal{A}\left(\mu\right)\left(1 + \epsilon_\mu^{\mathcal{A}}\right) = -\left(1 - \mathcal{A}\left(\mu\right)\right) + \epsilon_\mu^{\mathcal{A}} < 0.$$

Case 2. Some chained consumption.   If there are some chained expenditures:

$$C^s\left(B\right) \equiv \left(1 - \beta\right) B$$

$$E^w\left(B, G^s, G^x\right) \equiv 1 - \left(1 - \beta\right) B - G^x - G^s$$

$$C^w\left(B, \tilde{B}, G^s, G^x\right) \equiv S^w + X^w$$

$$S^w\left(B, \tilde{B}, G^s, G^x\right) \equiv \max\left\{\tilde{B} - B, 0\right\}$$

$$X^w\left(B, \tilde{B}, G^s, G^x\right) \equiv \frac{E^w - \max\left\{\tilde{B} - B, 0\right\}}{q}$$

$$q\left(B, \tilde{B}, G^s, G^x\right) \equiv \mathcal{A}^{-1}\left(\mu\right)$$

$$\mu\left(B, \tilde{B}, G^s, G^x\right) \equiv G^x + X^w q.$$

Rewriting the last three identities using $\mathcal{A}\left(\mu\right)$ instead of q we have

$$X^w \equiv \mathcal{A}\left(\mu\right)\left(E^w - \max\left\{\tilde{B} - B, 0\right\}\right)$$

$$\mu \equiv G^x + E^w - \max\left\{\tilde{B} - B, 0\right\}.$$

Substituting E^w we have, naturally,

$$\mu = 1 - \underbrace{\left(\left(1 - \beta\right) B + G^s + \max\left\{\tilde{B} - B, 0\right\}\right)}_{\text{spot exp.}}. \tag{49}$$

From here we obtain that:

$$\frac{dX^w}{dG^x} = -\mathcal{A}(\mu) + \mathcal{A}'(\mu)\frac{d\mu}{dG^x}\left(E^w - \max\left\{\tilde{B} - B, 0\right\}\right).$$

Since $\frac{d\mu}{dG^x} = \frac{dS^w}{dG^x} = 0$, we have that:

$$\frac{dC^w}{dG^x} = -\mathcal{A}(\mu).$$

Hence, the government multiplier for chained expenditures is:

$$\mathcal{M}^x\left(B, \tilde{B}\right) = \epsilon_\mu^{\mathcal{A}} < 1.$$

Next, observe that:

$$X^w \equiv \mathcal{A}(\mu)\left(1 - (1-\beta)B - G^x - G^s - \max\left\{\tilde{B} - B, 0\right\}\right).$$

Hence,

$$
\begin{aligned}
dX^w &= -\mathcal{A}(\mu)\,dG^s + \mathcal{A}'(\mu)\left(\frac{X^w}{\mathcal{A}(\mu)}\right)d\mu \\
&= -\mathcal{A}(\mu)\,dG^s + \mathcal{A}(\mu)\,\epsilon_\mu^{\mathcal{A}}\left(\frac{X^w}{\mathcal{A}(\mu)}/\mu\right)d\mu.
\end{aligned}
$$

The second line follows from:

$$\mathcal{A}(\mu)\mu \equiv G^x + X^w.$$

Also following this condition, we have that:

$$\mathcal{A}(\mu)\left(1 + \epsilon_\mu^{\mathcal{A}}\right)d\mu \equiv dX^w.$$

Combining the differentials evaluated at $G^x = 0$, we obtain:

$$d\mu = -dG^s.$$

Hence,

$$\frac{dC^w}{dG^s} = \frac{dX^w}{dG^s} = -\mathcal{A}(\mu)\left(1 + \epsilon_\mu^{\mathcal{A}}\right).$$

Following the relationship with the fiscal multiplier, we obtain:

$$\mathcal{M}^s\left(B, \tilde{B}\right) = 1 - \mathcal{A}(\mu) - \mathcal{A}(\mu)\,\epsilon_\mu^{\mathcal{A}}.$$

Summary.   We summarize the results:

$$\frac{dC^w}{dG^x} = \begin{cases} -1 & B < B^\star\left(\tilde{B}\right) \\ \\ -\mathcal{A}(\mu) & B > B^\star\left(\tilde{B}\right) \end{cases}$$

and

$$\frac{dC^w}{dG^s} = \begin{cases} -1 & B < B^\star\left(\tilde{B}\right) \\ \\ -\mathcal{A}(\mu)\left(1 + \epsilon_\mu^{\mathcal{A}}\right) & B > B^\star\left(\tilde{B}\right). \end{cases}$$

Finally, recall that the government multipliers relate to the change in consumption as follows:

$$\mathcal{M}^s\left(B, \tilde{B}\right) \equiv \frac{d\mathcal{Y}}{dG^s} = \frac{dC^w}{dG^s} + 1$$

and

$$\mathcal{M}^s\left(B, \tilde{B}\right) \equiv \frac{dC^w}{dG^x} + \mathcal{A}(\mu)$$

with:

$$\frac{d\mu}{dG^x} = \begin{cases} 1 & B < B^\star\left(\tilde{B}\right) \\ \\ 0 & B > B^\star\left(\tilde{B}\right). \end{cases}$$

Therefore, adding terms:

$$
\mathcal{M}^x \left(B, \tilde{B} \right) = \begin{cases} - \left(1 - \mathcal{A} \left(\mu \right) \left(1 + \epsilon_\mu^{\mathcal{A}} \right) \right) & B < B^\star \left(\tilde{B} \right) \\ \\ 0 & B > B^\star \left(\tilde{B} \right). \end{cases}
$$

Hence,

$$
\mathcal{M}^s \left(B, \tilde{B} \right) = \begin{cases} 0 & B < B^\star \left(\tilde{B} \right) \\ \\ 1 - \mathcal{A} \left(\mu \right) \left(1 + \epsilon_\mu^{\mathcal{A}} \right) & B > B^\star \left(\tilde{B} \right). \end{cases}
$$