



PERUVIAN ECONOMIC ASSOCIATION

A unified framework for pricing credit and equity
derivatives

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Working Paper No. 116, December 2017

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A UNIFIED FRAMEWORK FOR PRICING CREDIT AND EQUITY DERIVATIVES

Master Project

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June 2017

Abstract

This Master Project scrutinizes the underlying theoretical arguments within Bayraktar and Yang's (2011) (4) model and tests if it is robust with newer 2017 data. We demonstrate all the related strong mathematical foundations to understand their model. We also observe that the matching between the observed and estimated data is not as good as expected with the Ford Motor Company data yet being better for the SPX Index data. Thus we claim for the model to be improved with more recent research.

Introduction

In January 2017 the Economic Report for the President was published in the U.S. Congress (31). In its chapter 6, it discusses on the one hand the importance of the derivatives used to hedge against risks and on the other hand, their drawbacks such as concentrating risk rather than dispersing it and increasing exposure to risky assets. The fear of default risk led to a general unwillingness to enter into any additional transactions when the financial crisis exploded. The Dodd-Frank Act reformed the OTC market in derivatives in order to increase transparency to protect investors against a systemic risk generated by a new fear of default risk. Nevertheless, the new U.S. administration tries to dismantle the Dodd-Frank Act¹ so the importance of default risk arises not only as a theoretical issue but also as an economic policy-making concern.

The 2008-2009 financial crisis highlighted that default risk is a crucial component that links credit and equity derivatives and not considering this relationship could generate systemic risk². One way to model the relationship is given by *intensity-based approach* models. The element of surprise in this approach makes it attractive for modelling the default probability: at any instant there is a probability that an obligor will default and default is defined here as the first jump of a Cox Process with intensity λ which measures the **price of credit risk** or **probability of default**. As mentioned by Schoutens and Cariboni (2009) (33), the element of surprise is captured by the jump since sudden events in reality cause important changes on the view on the company's probability of default. Examples of this are a discovery of fraud, a default of a competitor, a terrorist attack, the end of a price bubble, etc.

Bayraktar and Yang (2011) (4) produce an intensity-based approach model in order to establish a unified framework that links credit and equity derivatives. The main model's feature is that the goodness of

¹<https://www.theguardian.com/commentisfree/2017/feb/06/dismantling-dodd-frank-donald-trump-gift-wall-street>

²Defined by Freixas et. al. (2015) (20) as the risk of threats to financial stability that impair the functioning of the financial system as a whole with significant adverse effects on the broader economy.

fitting for the Ford Motor Company's stock options³ and the SPX Index Options is remarkable. It outperforms previous models by showing that they have the best fitness regarding the implied volatility curves and surface, accounting for an implied volatility skew. Their model is free of arbitrage and the surface is generated under a risk-neutral measure. The importance of the implied volatility relies, as stated by Arratia (2014) (2), in that it is a forward looking measure since it represents the volatility of the asset into the future time until expiration of the option contract⁴.

In this Master Project we are going in detail through Bayraktar and Yang's (2011) (4) model in order to tackle not only its theoretical issues but also its empirical methodology. We want to see whether the goodness of fitting is robust by using 2017 data. The remainder of this Master Project is structured as follows. In the next section we review the related literature. In section 3 we analyze in detail the underlying theoretical features of the model used by Bayraktar and Yang (2011) (4) and formulate the hypothesis. In section 4 we explain the methodology of calibration of, among other parameters, the probability of default from bond data and the rest of the parameters from stock option data. In the section 5 we discuss our results and in the next section we develop our conclusions. The appendix contains figures and tables.

Review of the literature

The literature is vast considering the approach of the default time. As stated in Jeanblanc and Rutkowski (1999) (24), Schoutens and Cariboni (2009) (33) and Fouque et. al. (2011) (22), there are two main approaches to model default time: the *structural approach* and the *intensity-based, hazard rate or reduced form approach*. In the first approach default occurs when the firm's asset value hits a lower boundary. The main difference between this approach and the second one is that in the former the time of default is announced or predictable whereas in the latter it is not. This is, the default time is endogenous in the first approach and exogenous in the second one. Bayraktar and Yang (2011) (4) employ the second approach to develop their model.

The unified framework feature of these authors is highlighted by Chung and Kwok (2014) (14). They perform numerical valuations of the derivative prices using standard numerical integration quadrature or a Fast Fourier transform algorithm. Yamazaki (2013) (38) develops a jump to default exponential Lévy model in order to get a unified framework to explain the linkages between equity and credit derivatives, setting a variance gamma process and a Brownian motion as the driving factors of the model. Dyrssen et. al. (2014) (18) also stand out that companies' equity and debt are linked so a unified framework is crucial for risk management and hedging. In their theoretical working paper they find conditions ensuring that the option price at the default boundary coincides with the recovery payment and study the spatial convexity of the option price. For a broader view concerning the modeling of the interrelationships between equity and credit derivatives and its application, the thesis of Chun (2015) (15) is self-contained.

Bayraktar and Yang's (2011) (4) model is employed and modified by Takeyama, et. al. (2011) (36). These authors demonstrate that the model has to be changed with an appropriate description of the term structure by replacing the Vasicek model for interest rate with the Hull and White model. Instead of using prices, they work directly with the observed and estimated implied volatilities to obtain the parameters of their model and they refer this strategy as the reverse modeling of the probability of default. Choi and Sircar (2013) (13) extend the model assuming that the intensity is a function of the stock index variance and an idiosyncratic firm component, i.e., the intensity is endogenous. The index and its variance follow a stochastic volatility model and both the market factor and the idiosyncratic one are affine jump diffusions.

A more recent working paper developed by Chang and Orosi (2016) (10) questions the assumption

³The reason why this company was examined is given in the Appendix.

⁴It differs from historical volatility because the latter is estimated from past returns.

that states that the stock falls to 0 when the default occurs, because there could be a positive equity recovery at default that may be ignored because bankruptcy may not mean that the firm is insolvent or due to strategic bankruptcy. Consequently, the option implied default probability could be biased and their study take it into account allowing positive stock price at default.

Another direction through the default risk is given by Capponi and Larsson (2011) (8). They show and endogenous interaction between a stock and a defaultable bond by developing an equilibrium model that contains the relationship between cyclical properties of the default intensity, behaviour of investors and risk preferences. The probabilistic model is the same as the one used by Bayraktar and Yang (2011) (4) though they investigate how the size of the jump is affected by the risk aversion of the agents to propose measures of systemic risk.

Theoretical discussion and formulation of Hypothesis

The Model

The Model we are going to develop in this section is a Risk Neutral Model from Bayraktar and Yang (2011) (4). The probability space is $(\Omega, \mathcal{H}, \mathbb{P})$. On the one hand, we have five correlated standard Brownian Motions $\vec{W}_t = (W_t^0, W_t^1, W_t^2, W_t^3, W_t^4)$ characterized by a positive definite correlation matrix of the following form

$$\mathbb{E} \left[\vec{W}_t \vec{W}_t' \right] = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_3 & \rho_4 \\ \rho_1 & 1 & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_2 & \rho_{21} & 1 & \rho_{23} & \rho_{24} \\ \rho_3 & \rho_{31} & \rho_{32} & 1 & \rho_{34} \\ \rho_4 & \rho_{41} & \rho_{42} & \rho_{43} & 1 \end{bmatrix} \times t$$

with $\rho_{ij} = \rho_{ji} \forall i \neq j$ and $i \in \{0, 1, 2, 3, 4\}$, $j \in \{0, 1, 2, 3, 4\}$.

On the other hand we have a Cox Process (time-changed Poisson Process or doubly stochastic Poisson Process) $\tilde{N}_t \triangleq N \left(\int_0^t \lambda_s ds \right)$. The difference between the traditional Poisson Process and the Cox Process is that the latter's intensity is not constant, instead is a Stochastic Process λ_t (Schönbucher, 2003 (34)) so it entails a more general definition. Also, the intensity is never negative: $\lambda_t \geq 0$. Then, following Cont and Tankov (2004)(16), let $(\tau_i)_{i \geq 1}$ be a sequence of independent exponential random variables with intensity λ_t and $T_n = \sum_{i=1}^n \tau_i$. The process $(N_t, t \geq 0)$ defined by

$$\tilde{N}_t = \sum_{n \geq 1} \mathbb{1}_{t \geq T_n}$$

is a Cox Process with intensity or hazard rate λ_t , where $\mathbb{1}_{t \geq T_n} = \begin{cases} 1 & \text{if } T_n \leq t \\ 0 & \text{if } T_n > t > 0 \end{cases}$

This is, the indicator function establishes that if the jumps occur before t , then it is equal to 1, otherwise it is equal to 0. τ_i is the time of the jumps exhibited by the Cox Process. $\Delta \tilde{N}_t = \tilde{N}_t - \tilde{N}_{t-}$ denotes the jump size of \tilde{N}_t at time t , given that its trajectories are, with probability one, right continuous⁵ and piecewise constant. Let $\tau_1 = \tau$ the time of default. Thus, according to Bayraktar (2008) (3) Bayraktar and Yang (2011) (4) the time of the first jump of \tilde{N}_t is denoted by the stopping (or hitting) time:

$$\tau = \inf\{t \in \mathbb{R}_+ | \tilde{N}_t = 1\}$$

⁵Cont and Tankov (2004)(16) warn about this definition since it is not the same as left continuous. Intuitively, right continuous is called cadlag and it means "after" while left continuous is called caglad and it means "before". With left continuous sample paths, the jumps are predictable, a feature that does not happen when modeling defaults. However, with right continuous ones, jumps are sudden, unforeseeable events so the choice of right-continuity is natural.

where $n = 1$. τ is assumed to be exogenous: it is independent of asset prices and/or of the value of the firm. In line with Elliot et al. (2006) (19)

$$\tilde{N}_t = \mathbb{1}_{t \geq T_1} = \mathbb{1}_{\tau \leq t} = \begin{cases} 1 & \text{and } \Delta \tilde{N}_t = 1 & \text{if } T_1 = \tau_1 = \tau \leq t \\ 0 & \text{and } \Delta \tilde{N}_t = 0 & \text{if } T_1 = \tau_1 = \tau > t \end{cases}$$

Then λ_t is the hazard rate of τ . Following Shreve (2004) (35) and Schönbucher, 2003 (34), the countdown process $e^{-\int_0^t \lambda_s ds}$ is the expected time between the jumps. Additionally, as Cont and Tankov (2004)(16) point out, the first jump time on $[0, T]$ of the Cox Process is a uniformly distributed random variable U conditionally to $\tilde{N}_t = 1$. If we set $T = 1$, then we can rewrite the first jump of \tilde{N}_t as the first time the countdown process $e^{-\int_0^t \lambda_s ds}$ hits the level U as Schönbucher, 2003 (34)

$$\tau = \inf \left\{ t \in \mathbb{R}_+ \mid e^{-\int_0^t \lambda_s ds} \leq U \right\}$$

Let $A = -\ln(1 - U)$. $(1 - U)$ is also a uniform on $[0, 1]$ so also we can set $A = -\ln U$. Then $P(A \leq a) = P(-\ln(1 - U) \leq a) = P(1 - U \leq e^{-a}) = P(U \leq 1 - e^{-a}) = 1 - e^{-a}$ which is the cumulative distribution function of a unit exponential random variable. Thus, we can also rewrite the first jump of \tilde{N}_t as the first time the hazard process $e^{-\int_0^t \lambda_s ds}$ is greater or equal to the random level $a \sim \text{Exp}(1)$ as in Linetsky (2005) (26) and Carr and Linetsky (2006) (9) by taking logs to the left and right hand side of the former definition

$$\tau = \inf \left\{ t \in \mathbb{R}_+ \mid \int_0^t \lambda_s ds \geq a \right\}$$

This three definitions are equivalent and any of the three can be found in the literature. Therefore, we can follow Linetsky (2005) (26) so at the time of bankruptcy τ , the stock price jumps to the bankruptcy state Θ where it remains forever. Θ is called the *cemetery state* and is identified as $\Theta = 0$. The stock price subject to bankruptcy, or defaultable stock as called in Bayraktar (2008) (3), is modeled subject to a diffusion process $(\bar{X}_t, t \geq 0)$. Using Bayraktar and Yang (2011) (4) notation, $(X_t, t \geq 0)$ is the pre-bankruptcy stock price process so

$$\bar{X}_t = \begin{cases} X_t & \text{and } X_t = X_{t-} & \text{if } \tau > t \\ \Theta = 0 & \text{and } X_t \neq X_{t-} & \text{if } \tau \leq t \end{cases}$$

At the time of default, the stock price jumps down to 0. Let $\Gamma_t = \int_0^{t \wedge \tau} \lambda_u du$. The correct way of writing an stochastic differential equation is as modelled as Björk (2011) (7)

$$dX_t = r_t X_{t-} dt + \sigma_t X_{t-} dW_t^0 + X_{t-} d\Gamma_t + \mu X_{t-} d\tilde{N}_t$$

where X_{t-} can be replaced by X_t next to the dt and dW_t^0 terms since both the time t and the Brownian Motion W_t^0 do not exhibit jumps. Bielecki and Rutkowski (2002) (5) establish that $\int_0^{t \wedge \tau} \lambda_u du = \int_0^t \lambda_u \mathbb{1}_{\{\tau > t\}} du$. Thus, if \tilde{N}_t has a jump at time t , then $\Gamma_t = 0$ and the size of the jump of X , given that $\Delta \tilde{N}_t = 1$, is

$$\Delta X_t = \mu X_{t-}$$

so μ is the relative jump size of the stock price or the jump volatility of X . The sign of μ determines the sign of the jump: if $\mu > 0$ then all jumps are upwards whereas if $\mu < 0$. all jump are downwards. In particular we see that if $\mu = -1$ then, if there is a jump at t , we obtain $\Delta X_t = -X_{t-}$.

This is, if $\tau \leq t$, then $t \wedge \tau = \tau$, $\tilde{N}_t = 1$ and

$$X_t = X_{t-} + \Delta X_t = X_{t-} - X_{t-} = \Theta = 0$$

In other words, if $\mu = -1$, then the stock price will jump to zero at the first jump of \tilde{N} and the stock price will stay forever at the value zero, just as we have explained before. Consequently, Bayraktar and

Yang (2011) (4) modeled the stock price with an equivalent representation as the solution of the following stochastic differential equation

$$d\bar{X}_t = \bar{X}_t(r_t dt + \sigma_t dW_t^0 - dM_t)$$

with $\bar{X}_0 = x$. M_t is the compensated bankruptcy jump process as in Linetsky (2005) (26) and

$$M_t = \tilde{N}_t - \int_0^{t \wedge \tau} \lambda_u du$$

The discounted stock price $\widetilde{\bar{X}}_t = \bar{X}_t \exp(-\int_0^t r_s ds)$ is a Martingale ⁶ under the measure \mathbb{P} . Thereby,

$$d\widetilde{\bar{X}}_t = \widetilde{\bar{X}}_t(\sigma_t dW_t^0 - dM_t)$$

Moreover, by the definition above we can state that $\tilde{N}_t = \tilde{N}_{t \wedge \tau}$. Then we have:

$$\begin{aligned} \mathbb{E} \left[\tilde{N}_t - \int_0^{t \wedge \tau} \lambda_u du \middle| \mathcal{H}_s \right] &= \mathbb{E}[\tilde{N}_t - \tilde{N}_s + \tilde{N}_s | \mathcal{H}_s] - \mathbb{E} \left[\int_0^{s \wedge \tau} \lambda_u du \middle| \mathcal{H}_s \right] - \mathbb{E} \left[\int_{s \wedge \tau}^{t \wedge \tau} \lambda_u du \middle| \mathcal{H}_s \right] \\ \mathbb{E} \left[\tilde{N}_t - \int_0^{t \wedge \tau} \lambda_u du \middle| \mathcal{H}_s \right] &= \tilde{N}_s - \int_0^{s \wedge \tau} \lambda_u du + \mathbb{E}[\tilde{N}_{t \wedge \tau} - \tilde{N}_{s \wedge \tau} | \mathcal{H}_s] - \mathbb{E} \left[\int_{s \wedge \tau}^{t \wedge \tau} \lambda_u du \middle| \mathcal{H}_s \right] \end{aligned}$$

The last two terms are equal⁷ so M_t is a Martingale. The authors are using this equivalent representation and, as Björk (2011) (7) point it out, it is called the semimartingale decomposition of X dynamics. If $\tau > t$, then $t \wedge \tau = t$, $\tilde{N}_t = 0$, $\bar{X}_t = X_t$, $X_0 = x$ and⁸ $d \left(\int_0^t \lambda_u du \right) = \lambda_t dt$, then

$$dX_t = X_t((r_t + \lambda_t)dt + \sigma_t dW_t^0)$$

Applying Itô's fórmula to $f(t, X_t) = \ln(X_t)$ gives the solution for the pre-bankruptcy stock price, this is, the stock price without jumps

$$X_t = X_0 \exp \left\{ \int_0^t \sigma_s dW_s^0 + \int_0^t \left(r_s + \lambda_s - \frac{\sigma_s^2}{2} \right) ds \right\}$$

In this model, the interest rate r_t , the intensity λ_t and the volatility σ_t are stochastic processes.

Interest Rate

The interest rate r_t is modeled as an Ornstein-Uhlenbeck (Vasicek) process

$$dr_t = (\alpha - \beta r_t)dt + \eta dW_t^1$$

where $\alpha > 0$, $\beta > 0$ and $\eta > 0$ are parameters and $r_0 = r$.

Applying Itô's formula with $f(t, r_t) = e^{\beta t} r_t$ gives the result for the interest rate

$$r_t = \frac{\alpha}{\beta} + \left(r - \frac{\alpha}{\beta} \right) e^{-\beta t} + \eta e^{-\beta t} \int_0^t e^{\beta s} dW_s^1$$

As in Papageorgiu and Sircar (2008) (29), this interest rate has the favorable property that is mean-reverting around its long run mean $\frac{\alpha}{\beta}$.

⁶Because, as Privault (2016) (30) shows in the Itô multiplication table, $dt \times dt = dW_t^0 \times dt = d\tilde{N}_t \times dt = d \int_0^{t \wedge \tau} \lambda_u du \times dt = d \int_0^t \lambda_u \mathbb{1}_{\{\tau > t\}} du \times dt = \lambda_t \mathbb{1}_{\{\tau > t\}} dt \times dt = 0$.

⁷Because, as Fouque et. al. (2011) (22) point out, the increment $\tilde{N}_{t \wedge \tau} - \tilde{N}_{s \wedge \tau}$ is Poisson distributed with parameter $\int_{s \wedge \tau}^{t \wedge \tau} \lambda_u du$.

⁸By the Leibniz Integral Rule.

Intensity

The intensity $\lambda_t = f(Y_t, Z_t)$ is modeled as a strictly positive, bounded, smooth function that depends on two stochastic processes: Y_t and Z_t . We follow Fouque et. al. (2003) (21) for the description of both processes

$$dY_t = \frac{1}{\epsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}dW_t^2 \quad (1)$$

$$dZ_t = \delta c(Z_t)dt + \sqrt{\delta}g(Z_t)dW_t^3 \quad (2)$$

with $Y_0 = y$ and $Z_0 = z$.

The first factor Y_t driving the intensity λ_t is a fast mean reverting Ornstein-Uhlenbeck (Vasicek) process. $1/\epsilon$ is the rate of mean reversion of this process, around its long run mean m , with $\epsilon > 0$ being a small parameter which corresponds to the fast or short time scale of this process.

The second factor Z_t driving the intensity λ_t is a slowly varying diffusion process where $\delta > 0$ is a small parameter which corresponds to the slow or long time scale of this process. The authors assume that the functions c and g satisfy Lipschitz continuity and growth conditions so Z_t has a unique solution. As Fouque, et. al. (2011) (22) remark, their particular form does not play a role in the perturbation method analyzed after.

The solution for the first factor is obtained by following the same procedure as for the interest rate and starting by applying Itô's formula with $f(t, Y_t) = e^{\frac{1}{\epsilon}t}Y_t$. Then we get

$$Y_t = m + (y - m)e^{-\frac{1}{\epsilon}t} + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}e^{-\frac{1}{\epsilon}t} \int_0^t e^{\frac{1}{\epsilon}s} dW_s^2$$

Note that since ϵ is small, $\frac{1}{\epsilon}$ is high, so the smaller the ϵ , the faster the mean reversion around the long run mean m , i.e., as $\epsilon \rightarrow 0 \Rightarrow Y_t \rightarrow m$.

Similarly, the solution for the second factor is obtained applying Itô's formula with $f(t, Y_t) = e^{\delta t}Z_t$, getting

$$Z_t = ze^{-\delta t} + \delta e^{-\delta t} \int_0^t e^{\delta s}(Z_s + c(Z_s))ds + \sqrt{\delta}e^{-\delta t} \int_0^t e^{\delta s}g(Z_s)dW_s^3$$

For Z_t , is the other way around: the smaller the δ , the slower the mean reversion and as $\delta \rightarrow 0 \Rightarrow Z_t \rightarrow Z_0 = z$.

Volatility

The volatility is modeled as $\sigma_t = \sigma(\tilde{Y}_t)$ and its diffusion process is

$$d\tilde{Y}_t = \left(\frac{1}{\epsilon}(\tilde{m} - \tilde{Y}_t) - \frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}}\Lambda(\tilde{Y}_t) \right) dt + \frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}}dW_t^4$$

where Λ is a smooth, bounded function and it is the market price of volatility risk. The authors assume also that the function σ is bounded and smooth. Here, as for the intensity, the factor \tilde{Y}_t driving the volatility σ_t is fast mean reverting as Y_t . The solution is obtained with the same procedure and results as Y_t

$$\tilde{Y}_t = m + (\tilde{y} - m)e^{-\frac{1}{\epsilon}t} + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}e^{-\frac{1}{\epsilon}t} \left(\int_0^t e^{\frac{1}{\epsilon}s} dW_s^4 - \int_0^t e^{\frac{1}{\epsilon}s} \Lambda(\tilde{Y}_t) ds \right)$$

The fact that the volatility is a stochastic process makes the market to be incomplete as mentioned by Fouque et. al. (2011) (22): there is a whole family of equivalent martingale measures and derivatives securities cannot be perfectly hedged with just the stock and bond.

Equity and Credit Derivatives

For this subsection, it is important to understand the flows of information. The authors assume that $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is the filtration of the standard Brownian Motions \vec{W}_t . The filtration generated by \tilde{N}_t is $\mathbb{I} = \{\mathcal{I}_t, t \geq 0\}$. In addition, $\mathbb{G} = \{\mathcal{G}_t, t \geq 0\}$ is the enlargement of \mathbb{F} such that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{I}_t, t \geq 0$. This enlargement joins the information given by the standard Brownian Motions and the information given by the Cox Process. As Lando (1998) (25) points out, \mathcal{G}_t corresponds to knowing the evolution of the state variables up to time t and whether default has occurred or not. Notice that if $\tilde{N}_t = 0$, the relevant information of the model only depends on the standard Brownian Motions since there are no jumps, i.e. there is no default.

The authors price European Options and Bonds of the same company.

European Call Option

The price of a European Call Option with maturity T and strike Price K is given by

$$C(t; T, K) = \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) (\bar{X}_T - K)^+ \mathbb{1}_{\{\tau > T\}} \middle| \mathcal{G}_t \right] + \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) (\bar{X}_T - K)^+ \mathbb{1}_{\{\tau \leq T\}} \middle| \mathcal{G}_t \right]$$

This is, this price considers the expected present value of the payoff $(\bar{X}_T - K)^+$ when there is no default $\mathbb{1}_{\{\tau > T\}}$ and when there is default before maturity $\mathbb{1}_{\{\tau \leq T\}}$, given the information by \mathcal{G}_t . Yet, as we have seen above, $\bar{X}_T = X_T$ with no default and $\bar{X}_T = 0$ when there is default so the second term of the right hand side is always equal to zero. Thus we only have that the price is

$$C(t; T, K) = \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) (X_T - K)^+ \mathbb{1}_{\{\tau > T\}} \middle| \mathcal{G}_t \right]$$

We can simplify this last expression by putting all in terms of \mathcal{F}_t . This is due to Hull and White (1995) (23) who emphasize that the stochastic processes may change between when we move from the default free scenario to the vulnerable one. Therefore, we follow Lando's (1998) (25) procedure. First of all, we have that, given $t < T$

$$\mathbb{E}[\mathbb{1}_{\{\tau > T\}} | \mathcal{F}_T \vee \mathcal{I}_t] = \mathbb{E}[\mathbb{1}_{\{\tau > T\}} \times \mathbb{1}_{\{\tau > t\}} | \mathcal{F}_T \vee \mathcal{I}_t] = \mathbb{1}_{\{\tau > t\}} \mathbb{E}[\mathbb{1}_{\{\tau > T\}} | \mathcal{F}_T \vee \mathcal{I}_t] \quad (3)$$

the last equality is because $\mathbb{1}_{\{\tau > t\}}$ is $\mathcal{F} \vee \mathcal{I}$ -measurable, i.e., the economic agents will observe default when it happens or when it does not occur up to t as Elliot et al. (2006) (19) remark. Then, i) by using the definition of the conditional expected value of an indicator function, ii) the definition of conditional probability, and iii) $\{\tau > T\} \subset \{\tau > t\}$ we have

$$\mathbb{E}[\mathbb{1}_{\{\tau > T\}} | \mathcal{F}_T \vee \mathcal{I}_t] = \Pr(\tau > T | \mathcal{F}_T \vee \mathcal{I}_t) = \frac{\Pr(\{\tau > T\} \cap \{\tau > t\} | \mathcal{F}_T)}{\Pr(\tau > t | \mathcal{F}_T)} = \frac{\Pr(\{\tau > T\} | \mathcal{F}_T)}{\Pr(\tau > t | \mathcal{F}_T)}$$

The last two conditional probabilities in the numerator and denominator are called by Schönbucher (2003) (34) conditional *survival probabilities*. As Papageorgiu and Sircar (2008) (29) establish, this are the probabilities that the default time will be greater than t or T respectively, conditional on the information given by \mathcal{F}_t and \mathcal{F}_T respectively. Furthermore, following Schönbucher (2003) (34), for a Cox Process, the probability of $n = 0$ jumps is equal to:

$$\Pr(\tilde{N}_t = 0) = \mathbb{E} \left[\frac{1}{n!} \left(\int_0^t \lambda_s ds \right)^n \exp \left(- \int_0^t \lambda_s ds \right) \right] = \mathbb{E} \left[\exp \left(- \int_0^t \lambda_s ds \right) \right]$$

and given that $\int_0^t \lambda_s ds$ is \mathcal{F}_t -measurable

$$\Pr(\tilde{N}_t = 0 | \mathcal{F}_t) = \mathbb{E} \left[\exp \left(- \int_0^t \lambda_s ds \right) \middle| \mathcal{F}_t \right] = \exp \left(- \int_0^t \lambda_s ds \right)$$

This last result works for t and T . Therefore

$$\begin{aligned}\Pr(\{\tau > t\}|\mathcal{F}_t) &= \Pr(\tilde{N}_t = 0|\mathcal{F}_t) = \exp\left(-\int_0^t \lambda_s ds\right) \\ \Pr(\{\tau > T\}|\mathcal{F}_T) &= \Pr(\tilde{N}_T = 0|\mathcal{F}_T) = \exp\left(-\int_0^T \lambda_s ds\right)\end{aligned}$$

Hence

$$\frac{\Pr(\{\tau > T\}|\mathcal{F}_T)}{\Pr(\tau > t|\mathcal{F}_T)} = \frac{\exp\left(-\int_0^T \lambda_s ds\right)}{\exp\left(-\int_0^t \lambda_s ds\right)} = \exp\left(-\int_t^T \lambda_s ds\right)$$

Replacing this last term in (3) we get

$$\mathbb{E}[\mathbb{1}_{\{\tau > T\}}|\mathcal{F}_T \vee \mathcal{I}_t] = \mathbb{1}_{\{\tau > t\}}\mathbb{E}[\mathbb{1}_{\{\tau > T\}}|\mathcal{F}_T \vee \mathcal{I}_t] = \mathbb{1}_{\{\tau > t\}} \exp\left(-\int_t^T \lambda_s ds\right) \quad (4)$$

Now, the European call option $C(t; T, K)$ with maturity T and strike K can be simplified by using the law of total expectations

$$\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)(X_T - K)^+ \mathbb{1}_{\{\tau > T\}} \middle| \mathcal{G}_t\right] = \mathbb{E}\left[\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)(X_T - K)^+ \mathbb{1}_{\{\tau > T\}} \middle| \mathcal{F}_T \vee \mathcal{I}_t\right] \middle| \mathcal{G}_t\right]$$

And given that $\exp\left(-\int_t^T r_s ds\right)$ and $(X_T - K)^+$ are $\mathcal{F}_T \vee \mathcal{I}_t$ -measurable

$$= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)(X_T - K)^+ \mathbb{E}\left[\mathbb{1}_{\{\tau > T\}} \middle| \mathcal{F}_T \vee \mathcal{I}_t\right] \middle| \mathcal{G}_t\right]$$

Then, by applying (4) we obtain

$$= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)(X_T - K)^+ \mathbb{1}_{\{\tau > t\}} \exp\left(-\int_t^T \lambda_s ds\right) \middle| \mathcal{G}_t\right]$$

And given that $\mathbb{1}_{\{\tau > t\}}$ is \mathcal{G}_t -measurable we obtain

$$C(t; T, K) = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)(X_T - K)^+ \mathbb{1}_{\{\tau > T\}} \middle| \mathcal{G}_t\right] = \mathbb{1}_{\{\tau > t\}}\mathbb{E}\left[\exp\left(-\int_t^T (r_s + \lambda_s) ds\right)(X_T - K)^+ \middle| \mathcal{G}_t\right]$$

Nonetheless, we still have the conditional expectation with respect to \mathcal{G}_t . We want to replace it with \mathcal{F}_t . Given that the inner part of the last expectation is \mathcal{F}_t -measurable and that $\mathcal{F}_t \subseteq \mathcal{G}_t$, we apply the Tower Property of conditional expectations and we finally obtain Bayraktar and Yang (2011) (4) result

$$C(t; T, K) = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)(X_T - K)^+ \mathbb{1}_{\{\tau > T\}} \middle| \mathcal{G}_t\right] = \mathbb{1}_{\{\tau > t\}}\mathbb{E}\left[\exp\left(-\int_t^T (r_s + \lambda_s) ds\right)(X_T - K)^+ \middle| \mathcal{F}_t\right]$$

European Put Option

The price of a European Put Option with maturity T and strike Price K is given by

$$\text{Put}(t; T, K) = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)(K - \bar{X}_T)^+ \mathbb{1}_{\{\tau > T\}} \middle| \mathcal{G}_t\right] + \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)(K - \bar{X}_T)^+ \mathbb{1}_{\{\tau \leq T\}} \middle| \mathcal{G}_t\right]$$

This is, this price considers the expected present value of the payoff $(K - \bar{X}_T)^+$ when there is no default $\mathbb{1}_{\{\tau > T\}}$ and when there is default before maturity $\mathbb{1}_{\{\tau \leq T\}}$, given the information by \mathcal{G}_t . Yet, as we have seen above, $\bar{X}_T = X_T$ without default and $\bar{X}_T = 0$ when there is default so the price is

$$\text{Put}(t; T, K) = \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) (K - X_T)^+ \mathbb{1}_{\{\tau > T\}} \middle| \mathcal{G}_t \right] + \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) K \mathbb{1}_{\{\tau \leq T\}} \middle| \mathcal{G}_t \right]$$

This is different from the call option price since there is no recovery if the issuer goes bankrupt with a call option whereas there is recovery with a put option. The first term of the put option can be calculated exactly as the call option above, getting

$$\mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) (K - X_T)^+ \mathbb{1}_{\{\tau > T\}} \middle| \mathcal{G}_t \right] = \mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[\exp \left(- \int_t^T (r_s + \lambda_s) ds \right) (K - X_T)^+ \middle| \mathcal{F}_t \right]$$

In the second term notice that $\mathbb{1}_{\{\tau \leq T\}} = 1 - \mathbb{1}_{\{\tau > T\}}$ so replacing this in the second term we have

$$\begin{aligned} \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) K \mathbb{1}_{\{\tau \leq T\}} \middle| \mathcal{G}_t \right] &= K \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) (1 - \mathbb{1}_{\{\tau > T\}}) \middle| \mathcal{G}_t \right] \\ &= K \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) \middle| \mathcal{G}_t \right] - K \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) \mathbb{1}_{\{\tau > T\}} \middle| \mathcal{G}_t \right] \end{aligned}$$

Nonetheless, we still have the conditional expectation with respect to \mathcal{G}_t . We want to replace it with \mathcal{F}_t . For the positive term we recall that, as in Bielecki et. al. (2006) (6), $B(t; T) = \exp \left(- \int_t^T r_s ds \right)$ is the price of a default-free zero-coupon bond with maturity T at time t by definition, that is, on $\tau > t$. This is defined with a deterministic interest rate, but here we are dealing with a stochastic one so we have to work with conditional expectations. Therefore, applying the Tower Property, since $\mathcal{F}_t \subseteq \mathcal{G}_t$, and knowing that $\mathbb{1}_{\{\tau > t\}}$ is \mathcal{F}_t -measurable we get

$$K \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) \middle| \mathcal{G}_t \right] = K \mathbb{E} \left[\mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) \mathbb{1}_{\{\tau > t\}} \middle| \mathcal{G}_t \right] \middle| \mathcal{F}_t \right] = \mathbb{1}_{\{\tau > t\}} K \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) \middle| \mathcal{F}_t \right]$$

For the negative term we follow the same procedure as for the call option⁹ and then we finally obtain the following result¹⁰

$$\begin{aligned} \text{Put}(t; T, K) &= \mathbb{1}_{\{\tau > t\}} \left\{ \mathbb{E} \left[\exp \left(- \int_t^T (r_s + \lambda_s) ds \right) (K - X_T)^+ \middle| \mathcal{F}_t \right] \right. \\ &\quad \left. + K \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) \middle| \mathcal{F}_t \right] - K \mathbb{E} \left[\exp \left(- \int_t^T (r_s + \lambda_s) ds \right) \middle| \mathcal{F}_t \right] \right\} \end{aligned}$$

Put-Call Parity

The Put-Call Parity defines the relationship between an European Call Option and an European Put Option, both with identical strike price K and the same underlying asset. With the above expressions of both options

⁹Instead of $(\bar{X}_T - K)^+$ we have here a payoff equal to 1.

¹⁰This is different from Linetsky (2005) (26) where the put pricing formula consists on two parts: the present value of the put payoff given no bankruptcy (the first expected value in our expression, multiplied by the indicator function) and the present value of the recovery in the event of bankruptcy (the two last terms in our expression, without the indicator function). This difference appears because his model is developed with a constant interest rate, uncorrelated with the stochastic intensity.

we can obtain the parity

$$C(t; T, K) - \text{Put}(t; T, K) = \mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[\exp \left(- \int_t^T (r_s + \lambda_s) ds \right) (X_T - K)^+ \middle| \mathcal{F}_t \right] \\ - \mathbb{1}_{\{\tau > t\}} \left\{ \mathbb{E} \left[\exp \left(- \int_t^T (r_s + \lambda_s) ds \right) (K - X_T)^+ \middle| \mathcal{F}_t \right] \right. \\ \left. - K \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) \middle| \mathcal{F}_t \right] + K \mathbb{E} \left[\exp \left(- \int_t^T (r_s + \lambda_s) ds \right) \middle| \mathcal{F}_t \right] \right\}$$

$$C(t; T, K) - \text{Put}(t; T, K) = \mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[\exp \left(- \int_t^T (r_s + \lambda_s) ds \right) ((X_T - K)^+ - (K - X_T)^+) \middle| \mathcal{F}_t \right] \\ - \mathbb{1}_{\{\tau > t\}} \left\{ K \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) \middle| \mathcal{F}_t \right] + K \mathbb{E} \left[\exp \left(- \int_t^T (r_s + \lambda_s) ds \right) \middle| \mathcal{F}_t \right] \right\}$$

And since $(X_T - K)^+ - (K - X_T)^+ = X_T - K$, we get

$$C(t; T, K) - \text{Put}(t; T, K) = \mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[\exp \left(- \int_t^T (r_s + \lambda_s) ds \right) (X_T - K) \middle| \mathcal{F}_t \right] \\ - \mathbb{1}_{\{\tau > t\}} \left\{ K \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) \middle| \mathcal{F}_t \right] + K \mathbb{E} \left[\exp \left(- \int_t^T (r_s + \lambda_s) ds \right) \middle| \mathcal{F}_t \right] \right\}$$

By subtracting properly, we get

$$C(t; T, K) - \text{Put}(t; T, K) = \mathbb{1}_{\{\tau > t\}} \left\{ \mathbb{E} \left[\exp \left(- \int_t^T (r_s + \lambda_s) ds \right) X_T \middle| \mathcal{F}_t \right] - K \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) \middle| \mathcal{F}_t \right] \right\}$$

Finally, since $\mathbb{E} \left[\exp \left(- \int_t^T (r_s + \lambda_s) ds \right) X_T \middle| \mathcal{F}_t \right]$ is the present value of the stock price discounted by the interest rate adjusted by the intensity given the information up to t (this is, it is equal to X_t) and $B(t; T) = \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) \middle| \mathcal{F}_t \right]$ is the risk-free zero coupon bond with maturity T at time t , we obtain the Put-Call Parity associated to the model

$$C(t; T, K) - \text{Put}(t; T, K) = \mathbb{1}_{\{\tau > t\}} [X_t - KB(t; T)]$$

Defaultable Bond

The defaultable bond has maturity T and par value of 1 dollar. If the issuer company defaults prior to maturity, the holder of the bond recovers a constant fraction $(1 - l)$ of the pre-default value, with $l \in [0, 1]$. l is the loss given default. The price of this bond, similar in Papageorgiu and Sircar (2008) (29), is

$$B^c(t; T) = \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) \mathbb{1}_{\{\tau > T\}} + \exp \left(- \int_t^\tau r_s ds \right) \mathbb{1}_{\{\tau \leq T\}} (1 - l) B^c(\tau -; T) \middle| \mathcal{G}_t \right]$$

Duffie and Singleton (1999) (17) give an heuristic explanation of the following equality, which is the same as in Lando (1998) (25) and Bayraktar and Yang (2011) (4) result on $\tau > t$

$$B^c(t; T) = \mathbb{E} \left[\exp \left(- \int_t^T r_s ds \right) \mathbb{1}_{\{\tau > T\}} + \exp \left(- \int_t^\tau r_s ds \right) \mathbb{1}_{\{\tau \leq T\}} (1 - l) B^c(\tau -; T) \middle| \mathcal{G}_t \right] \\ = \mathbb{1}_{\{\tau > t\}} \mathbb{E} \left[\exp \left(- \int_t^T (r_s + l\lambda_s) ds \right) \middle| \mathcal{F}_t \right]$$

They state that the market value of the defaultable bond at time t obeys the last equality. The r.h.s. of this equality shows that in the event of no default, the contingent claim (the bond) pays 1 at time T . This claim may be priced as if it were default-free by replacing the usual short-term interest rate process r_t with the default-adjusted short-rate process $r_t + l\lambda_t$ where $l\lambda_t$ is the mean-loss rate. Discounting at the default-adjusted short-rate process therefore accounts on $\tau > t$ for both the probability and timing of default, as well as for the effect of losses on default.

The expressions obtained for the Equity and Credit Derivatives cannot be solved analytically so we have to use an asymptotic expansion to get the results.

Pricing Equation

Notice that up to this section, all the expressions for the state variables of the Model have been turned only dependent of a drift and standard Brownian Motion. With this in mind, let $P^{\epsilon, \delta}$ be the hedging portfolio of a payoff $h(X_T)$ denoted by

$$P^{\epsilon, \delta}(t, X_t, r_t, Y_t, \tilde{Y}_t, Z_t) = \mathbb{E} \left[\exp \left(- \int_t^T (r_s + l\lambda_s) ds \right) h(X_T) \middle| \mathcal{F}_t \right] \quad (5)$$

If $l = 1$ and $h(X_T) = (X_T - K)^+$, then $P^{\epsilon, \delta}$ is the price of a Call Option on a defaultable stock. Also, if $h(X_T) = 1$, then $P^{\epsilon, \delta}$ is the price of a defaultable bond.

To characterize $P^{\epsilon, \delta}$, we have to use the Multidimensional Version of the Feynman-Kac Theorem, as in Nualart (1997) (28). We start with the dynamics of the stochastic processes (vector) described within the Model

$$d\mathbf{s}_t = u(\mathbf{s}_t, t)dt + v(\mathbf{s}_t, t)d\mathbf{B}_t$$

where \mathbf{s}_t , \mathbf{B}_t is a vector of dimension m of uncorrelated standard Brownian Motions and $u(\mathbf{s}_t, t)$ are each vectors of dimension n and $v(\mathbf{s}_t, t)$ is a matrix of size $n \times m$. $u(\mathbf{s}_t, t)$ is the drift and $v(\mathbf{s}_t, t)$ is the volatility of \mathbf{s}_t . In other words,

$$d \begin{pmatrix} s_{1t} \\ s_{2t} \\ \vdots \\ s_{nt} \end{pmatrix} = \begin{pmatrix} u_{1t} \\ u_{2t} \\ \vdots \\ u_{nt} \end{pmatrix} dt + \begin{pmatrix} v_{11t} & v_{12t} & \cdots & v_{1mt} \\ v_{21t} & v_{22t} & \cdots & v_{2mt} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1t} & v_{n2t} & \cdots & v_{nmt} \end{pmatrix} \times d \begin{pmatrix} B_{1t} \\ B_{2t} \\ \vdots \\ B_{mt} \end{pmatrix}$$

We can associate to the diffusion process a second order differential operator or generator process that is

$$\mathcal{A} = \sum_{i=1}^n u_i \frac{\partial}{\partial s_i} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (vv')_{ij} \frac{\partial^2}{\partial s_i \partial s_j}$$

where $u_i = u_i(\mathbf{s}_t, t)$, $v_i = v_i(\mathbf{s}_t, t)$ and $(vv')_{ij}$ is element ij of the matrix vv' of size $n \times n$. The matrix vv' is a symmetric and non negative definite matrix.

The Feynman-Kac Theorem states that the partial differential equation (PDE) in $V(\mathbf{s}_t, t)$, the hedging portfolio, is given by

$$\frac{\partial V}{\partial t} + \mathcal{A}V(\mathbf{s}_t, t) - r(\mathbf{s}_t, t)V(\mathbf{s}_t, t) = 0$$

and with a boundary condition the solution can be found.

On the following we avoid the sub-index t and capital letters to simplify the notation. In the model $n = m = 5$, $\mathbf{s} = (x, r, y, \tilde{y}, z)'$, $\mathbf{u} = (r + f(y, z), \alpha - \beta r, \frac{1}{\epsilon}(m - y), \frac{1}{\epsilon}(\tilde{m} - \tilde{y}) - \frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}}\Lambda(\tilde{y}), \delta c(z))'$ where $\lambda = f(y, z)$

and¹¹

$$v(\mathbf{s}) = \begin{bmatrix} \sigma(\tilde{y})x & 0 & 0 & 0 & 0 \\ \eta\rho_1 & \eta\sqrt{1-\rho_1^2} & 0 & 0 & 0 \\ \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\rho_2 & \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\textcircled{A} & \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\textcircled{B} & 0 & 0 \\ \frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}}\rho_4 & \frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}}\textcircled{C} & \frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}}\textcircled{D} & \frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}}\textcircled{E} & 0 \\ \sqrt{\delta}g(z)\rho_3 & \sqrt{\delta}g(z)\textcircled{F} & \sqrt{\delta}g(z)\textcircled{G} & \sqrt{\delta}g(z)\textcircled{H} & \sqrt{\delta}g(z)\textcircled{I} \end{bmatrix}$$

where $\textcircled{A} = \frac{\rho_{12} - \rho_1\rho_2}{\sqrt{1-\rho_1^2}}$, $\textcircled{B} = \sqrt{1-\rho_2^2 - \textcircled{A}^2}$, $\textcircled{C} = \frac{\rho_{14} - \rho_1\rho_4}{\sqrt{1-\rho_1^2}}$, $\textcircled{D} = \frac{\rho_{24} - \rho_1\rho_4 - \textcircled{A}\textcircled{C}}{\textcircled{B}}$,

$$\textcircled{E} = \sqrt{1-\rho_4^2 - \textcircled{C}^2 - \textcircled{D}^2}, \textcircled{F} = \frac{\rho_{13} - \rho_1\rho_3}{\sqrt{1-\rho_1^2}}, \textcircled{G} = \frac{\rho_{23} - \rho_2\rho_3 - (\rho_{12} - \rho_1\rho_2)(\rho_{13} - \rho_1\rho_3)}{\textcircled{B}},$$

$$\textcircled{H} = \frac{\rho_{34} - \rho_3\rho_4 - \textcircled{C}\textcircled{F} - \textcircled{D}\textcircled{G}}{\textcircled{E}}, \textcircled{I} = \sqrt{1-\rho_3^2 - \textcircled{F}^2 - \textcircled{G}^2 - \textcircled{H}^2} \text{ Then we have}$$

$$v(\mathbf{s})v(\mathbf{s})' = \begin{bmatrix} \sigma(\tilde{y})^2x^2 & \sigma(\tilde{y})x\eta\rho_1 & \sigma(\tilde{y})x\frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\rho_2 & \sigma(\tilde{y})x\frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}}\rho_4 & \sigma(\tilde{y})x\sqrt{\delta}g(z)\rho_3 \\ \sigma(\tilde{y})x\eta\rho_1 & \eta^2 & \eta\frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\rho_{12} & \eta\frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}}\rho_{14} & \eta\sqrt{\delta}g(z)\rho_{13} \\ \sigma(\tilde{y})x\frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\rho_2 & \eta\frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\rho_{12} & \frac{2\nu^2}{\epsilon} & \frac{2\nu\tilde{\nu}}{\epsilon}\rho_{24} & \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\sqrt{\delta}g(z)\rho_{23} \\ \sigma(\tilde{y})x\frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}}\rho_4 & \eta\frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}}\rho_{14} & \frac{2\nu\tilde{\nu}}{\epsilon}\rho_{24} & \frac{2\tilde{\nu}^2}{\epsilon} & \frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}}\sqrt{\delta}g(z)\rho_{34} \\ \sigma(\tilde{y})x\sqrt{\delta}g(z)\rho_3 & \eta\sqrt{\delta}g(z)\rho_{13} & \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\sqrt{\delta}g(z)\rho_{23} & \frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}}\sqrt{\delta}g(z)\rho_{34} & \delta g(z)^2 \end{bmatrix}$$

Therefore, using \mathbf{u} and $v(\mathbf{s})v(\mathbf{s})'$ we obtain

$$\begin{aligned} \mathcal{A} = & (r + f(y, z))x \frac{\partial}{\partial x} + (\alpha - \beta r) \frac{\partial}{\partial r} + \frac{1}{\epsilon}(m - y) \frac{\partial}{\partial y} + \left(\frac{1}{\epsilon}(\tilde{m} - \tilde{y}) - \frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}}\Lambda(\tilde{y}) \right) \frac{\partial}{\partial \tilde{y}} + \delta c(z) \frac{\partial}{\partial z} + \\ & \frac{1}{2}\sigma(\tilde{y})^2x^2 \frac{\partial^2}{\partial x^2} + \frac{1}{2}\eta^2 \frac{\partial^2}{\partial r^2} + \frac{\nu^2}{\epsilon} \frac{\partial^2}{\partial y^2} + \frac{\tilde{\nu}^2}{\epsilon} \frac{\partial^2}{\partial \tilde{y}^2} + \frac{1}{2}\delta g(z)^2 \frac{\partial^2}{\partial z^2} + \sigma(\tilde{y})x\eta\rho_1 \frac{\partial^2}{\partial x\partial r} + \sigma(\tilde{y})x\frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\rho_2 \frac{\partial^2}{\partial x\partial y} + \\ & \sigma(\tilde{y})x\frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}}\rho_4 \frac{\partial^2}{\partial x\partial \tilde{y}} + \sigma(\tilde{y})x\sqrt{\delta}g(z)\rho_3 \frac{\partial^2}{\partial x\partial z} + \eta\frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\rho_{12} \frac{\partial^2}{\partial r\partial y} + \eta\frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}}\rho_{14} \frac{\partial^2}{\partial r\partial \tilde{y}} + \eta\sqrt{\delta}g(z)\rho_{13} \frac{\partial^2}{\partial r\partial z} + \\ & \frac{2\nu\tilde{\nu}}{\epsilon}\rho_{24} \frac{\partial^2}{\partial y\partial \tilde{y}} + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\sqrt{\delta}g(z)\rho_{23} \frac{\partial^2}{\partial y\partial z} + \frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}}\sqrt{\delta}g(z)\rho_{34} \frac{\partial^2}{\partial \tilde{y}\partial z} \end{aligned}$$

Having $V = P^{\epsilon, \delta}$, it can be characterized as a solution of both the PDE

$$\frac{\partial P^{\epsilon, \delta}}{\partial t} + \mathcal{A}P^{\epsilon, \delta} - (r + lf(y, z))P^{\epsilon, \delta} = 0 \quad (6)$$

and the boundary condition:

$$P^{\epsilon, \delta}(T, x, r, y, \tilde{y}, z) = h(x)$$

Notice that the PDE in this case is employing the default-adjusted short-rate $r + lf(y, z)$ since this rate is using for discount. Consequently, Bayraktar and Yang (2011) (4) factorize the components of (6) in

¹¹The correlated Standard Brownian Motions above have been expressed in terms of uncorrelated Standard Brownian Motions to get $v(\mathbf{s})$. Also, we started from the matrix $v(\mathbf{s})v(\mathbf{s})'$ below and then applied a Cholesky Decomposition to obtain $v(\mathbf{s})$.

the partial differential operator $\mathcal{L}^{\epsilon,\delta}$ in terms of the parameters ϵ and δ as

$$\mathcal{L}^{\epsilon,\delta} \triangleq \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3 \quad (7)$$

where $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ are the factors defined by the authors, obtaining

$$\mathcal{L}^{\epsilon,\delta}P^{\epsilon,\delta}(t, x, r, y, \tilde{y}, z) = 0 \quad (8)$$

and the boundary condition:

$$P^{\epsilon,\delta}(T, x, r, y, \tilde{y}, z) = h(x)$$

Asymptotic Expansion

According to Malhalm (2005) (27), although we cannot get analytic results for the Equity and Credit Derivatives prices, we can have approximate analytic answers. We have to use Perturbation Methods which try to exploit the smallness of an inherent parameter in order to achieve this.

Observe that in (7) the operator terms that are associated with the parameter ϵ are diverging when $\epsilon \rightarrow 0$ while the terms associated with only the parameter δ are small when $\delta \rightarrow 0$. These give rise respectively to a *singular* perturbation problem and a *regular* perturbation problem. Nevertheless, the model has been re-scaled so the *singular* perturbation problem can be solved as a *regular* perturbation problem, changing the terms to get equated to 0 as we see below.

Expansion in the Slow-Scale

First of all, Bayraktar and Yang (2011) (4) apply an Expansion Method of $P^{\epsilon,\delta}$ in powers of $\sqrt{\delta}$

$$P^{\epsilon,\delta} = P_0^\epsilon + \sqrt{\delta}P_1^\epsilon + \delta P_2^\epsilon + \dots = \sum_{i=0}^{\infty} (\sqrt{\delta})^i P_i^\epsilon$$

By inserting this only considering it up to the first correction P_1^ϵ in (8) as in Fouque et. al. (2003) (21) and following Malhalm (2005) (27) we get

$$\begin{aligned} 0 = & \frac{1}{\epsilon}\mathcal{L}_0(P_0^\epsilon + \sqrt{\delta}P_1^\epsilon) + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1(P_0^\epsilon + \sqrt{\delta}P_1^\epsilon) + \mathcal{L}_2(P_0^\epsilon + \sqrt{\delta}P_1^\epsilon) + \\ & \sqrt{\delta}\mathcal{M}_1(P_0^\epsilon + \sqrt{\delta}P_1^\epsilon) + \delta\mathcal{M}_2(P_0^\epsilon + \sqrt{\delta}P_1^\epsilon) + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3(P_0^\epsilon + \sqrt{\delta}P_1^\epsilon) \end{aligned}$$

Now we have to get the $(\sqrt{\delta})^0$ and $(\sqrt{\delta})^1$ terms and equate them to 0

$$(\sqrt{\delta})^0 : \left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 \right) P_0^{\epsilon,\delta} = 0 \quad (9)$$

$$(\sqrt{\delta})^1 : \left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 \right) P_1^{\epsilon,\delta} + \left(\mathcal{M}_1 + \frac{1}{\sqrt{\epsilon}}\mathcal{M}_3 \right) P_0^{\epsilon,\delta} = 0 \quad (10)$$

and since $P_0^{\epsilon,\delta}$ is the solution within the Perturbation Methods (if $\sqrt{\delta} \rightarrow 0$, then $P^{\epsilon,\delta} = P_0^\epsilon$), it satisfies the boundary condition

$$P_0^{\epsilon,\delta}(T, x, r, y, \tilde{y}, z) = h(x)$$

so $P_1^{\epsilon,\delta}$ does not

$$P_1^{\epsilon,\delta}(T, x, r, y, \tilde{y}, z) = 0$$

Expansion in the Fast-Scale

Secondly, Bayraktar and Yang (2011) (4) apply an Expansion Method of P_0^ϵ and P_1^ϵ in powers of $\sqrt{\epsilon}$

$$P_0^\epsilon = P_{0,0} + \sqrt{\epsilon}P_{1,0} + \epsilon P_{2,0} + \cdots = \sum_{i=0}^{\infty} (\sqrt{\epsilon})^i P_{i,0}$$

$$P_1^\epsilon = P_{0,1} + \sqrt{\epsilon}P_{1,1} + \epsilon P_{2,1} + \cdots = \sum_{i=0}^{\infty} (\sqrt{\epsilon})^i P_{0,i}$$

Thereby, the complete expression for the pricing equation is

$$P^{\epsilon,\delta} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (\sqrt{\epsilon})^i (\sqrt{\delta})^j P_{i,j} \quad (11)$$

The leading term will simply be denoted by $P_0 = P_{0,0}$.

By inserting P_0^ϵ only considering it up to the third correction $P_{3,0}$ in (9) and following the same steps as above we get

$$0 = \frac{1}{\epsilon} \mathcal{L}_0(P_0 + \sqrt{\epsilon}P_{1,0} + \epsilon P_{2,0} + \epsilon^{3/2}P_{3,0}) + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1(P_0 + \sqrt{\epsilon}P_{1,0} + \epsilon P_{2,0} + \epsilon^{3/2}P_{3,0}) + \mathcal{L}_2(P_0 + \sqrt{\epsilon}P_{1,0} + \epsilon P_{2,0} + \epsilon^{3/2}P_{3,0})$$

Now we have to get the $(\sqrt{\epsilon})^{-2}$, $(\sqrt{\epsilon})^{-1}$, $(\sqrt{\epsilon})^0$ and $(\sqrt{\epsilon})^1$ terms and equate them to 0

$$(\sqrt{\epsilon})^{-2} : \mathcal{L}_0 P_0 = 0 \quad (12)$$

$$(\sqrt{\epsilon})^{-1} : \mathcal{L}_0 P_{1,0} + \mathcal{L}_1 P_0 = 0 \quad (13)$$

$$(\sqrt{\epsilon})^0 : \mathcal{L}_0 P_{2,0} + \mathcal{L}_1 P_{1,0} + \mathcal{L}_2 P_0 = 0 \quad (14)$$

$$(\sqrt{\epsilon})^1 : \mathcal{L}_0 P_{3,0} + \mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0} = 0 \quad (15)$$

Recall that \mathcal{L}_0 and \mathcal{L}_1 contain derivatives with respect to y and \tilde{y} so these are two ordinary differential equations. Therefore, by (12) we have that $\mathcal{L}_1 P_0 = 0$ in (13) so $\mathcal{L}_0 P_{1,0} = 0$. This is, P_0 and $P_{1,0}$ do not depend on y and \tilde{y} . Thus, $P_0 = P_0(t, x, r, z)$ and $P_{1,0} = P_{1,0}(t, x, r, z)$. For this last reason, $\mathcal{L}_1 P_{1,0} = 0$ and (14) turns to be

$$\mathcal{L}_0 P_{2,0} + \mathcal{L}_2 P_0 = 0$$

$$\mathcal{L}_0 P_{2,0} = -\mathcal{L}_2 P_0$$

The latter is a Poisson Equation for $P_{2,0}$ with respect to y and \tilde{y} , i.e., $P_{2,0}$ is the unknown. Therefore, $\mathcal{L}_2 P_0$ must satisfy the *solvability* or *centering* condition

$$\langle \mathcal{L}_2 P_0 \rangle = 0$$

where $\langle \cdot \rangle$ denotes the averaging with respect to the invariant distribution¹² of (y, \tilde{y}) , whose density is given by a bi-variate normal distribution function¹³, since the standard Brownian Motions W_t^2 and W_t^4 are correlated. Since P_0 does not depend on y and \tilde{y} we have

$$\langle \mathcal{L}_2 P_0 \rangle = \int \mathcal{L}_2 P_0 \Psi(y, \tilde{y}) dy d\tilde{y} = P_0 \int \mathcal{L}_2 \Psi(y, \tilde{y}) dy d\tilde{y} = \langle \mathcal{L}_2 \rangle P_0 = 0 \quad (16)$$

¹²An invariant distribution is also called a stationary or equilibrium distribution, i.e, a distribution that starts at $t = 0$ remains the same when $t > 0$. In this case, it is invariant with respect to $\frac{1}{\epsilon}$.

¹³Its density is captured in expression (3.11) by Bayraktar and Yang (2011) (4). However, there is a typo since it has to be divided by $\sqrt{1 - \rho_{2,4}^2}$

Recall that \mathcal{L}_2 contains derivatives with respect to t , x , and r and also contain functions of y and \tilde{y} . Thus, this last equality intuitively means that in order to find the solution of $P_{2,0}$, all the first order, second order and cross derivatives (changes) of $P_0(t, x, r, z)$ with respect to t , x , and r must, on average, do not change with movements on y and \tilde{y} .¹⁴

Using the Poisson Equation and subtracting (16) we can deduce that

$$P_{2,0} = -\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle)P_0 \quad (17)$$

The expression (15) is another Poisson Equation for $P_{3,0}$ with respect to y and \tilde{y} , i.e., $P_{3,0}$ is the unknown. The *solvability* or *centering* condition for this equation requires, employing (17), that

$$\langle \mathcal{L}_2 P_{1,0} \rangle = -\langle \mathcal{L}_1 P_{2,0} \rangle = \langle \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle P_0$$

which along the terminal condition $P_{1,0}(T, x, r, z) = 0$ define completely $P_{1,0}$. The *solvability* condition (16) and the terminal condition $P_0(T, x, r, z) = h(x)$ define completely the leading order term P_0 .

Finally, by inserting P_1^ϵ in (10) only considering it up to the second correction $P_{2,1}$, we get

$$\begin{aligned} 0 = & \frac{1}{\epsilon} \mathcal{L}_0(P_{0,1} + \sqrt{\epsilon}P_{1,1} + \epsilon P_{2,1}) + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1(P_{0,1} + \sqrt{\epsilon}P_{1,1} + \epsilon P_{2,1}) + \mathcal{L}_2(P_{0,1} + \sqrt{\epsilon}P_{1,1} + \epsilon P_{2,1}) \\ & + \mathcal{M}_1(P_0 + \sqrt{\epsilon}P_{1,0} + \epsilon P_{2,0}) + \frac{1}{\sqrt{\epsilon}} \mathcal{M}_3(P_0 + \sqrt{\epsilon}P_{1,0} + \epsilon P_{2,0}) \end{aligned}$$

Now we have to get the $(\sqrt{\epsilon})^{-2}$, $(\sqrt{\epsilon})^{-1}$ and $(\sqrt{\epsilon})^0$ terms and equate them to 0

$$(\sqrt{\epsilon})^{-2} : \mathcal{L}_0 P_{0,1} = 0 \quad (18)$$

$$(\sqrt{\epsilon})^{-1} : \mathcal{L}_0 P_{1,1} + \mathcal{L}_1 P_{0,1} + \mathcal{M}_3 P_0 = 0 \quad (19)$$

$$(\sqrt{\epsilon})^0 : \mathcal{L}_0 P_{2,1} + \mathcal{L}_1 P_{1,1} + \mathcal{L}_2 P_{0,1} + \mathcal{M}_1 P_0 + \mathcal{M}_3 P_{1,0} = 0 \quad (20)$$

Recall that \mathcal{M}_3 contain derivatives with respect to y and \tilde{y} so $\mathcal{M}_3 P_0 = \mathcal{M}_3 P_{1,0} = 0$. Thereby, by (18) we have that $\mathcal{L}_1 P_{0,1} = 0$ in (19) so $\mathcal{L}_0 P_{1,1} = 0$. This is, $P_{0,1}$ and $P_{1,1}$ do not depend on y and \tilde{y} . Thus, $P_{0,1} = P_{0,1}(t, x, r, z)$ and $P_{1,1} = P_{1,1}(t, x, r, z)$. For this reason, $\mathcal{L}_1 P_{1,1} = 0$ and (20) turns to be

$$\mathcal{L}_0 P_{2,1} + \mathcal{L}_2 P_{0,1} + \mathcal{M}_1 P_0 = 0$$

$$\mathcal{L}_0 P_{2,1} = -\mathcal{L}_2 P_{0,1} - \mathcal{M}_1 P_0$$

The latter is a Poisson equation for $P_{2,1}$ with respect to y and \tilde{y} , i.e., $P_{2,1}$ is the unknown. Therefore, $\mathcal{L}_2 P_{0,1} + \mathcal{M}_1 P_0$ must satisfy the *solvability* or *centering* condition

$$\langle \mathcal{L}_2 P_{0,1} + \mathcal{M}_1 P_0 \rangle = 0$$

and since P_0 and $P_{0,1}$ do not depend on y and \tilde{y}

$$\langle \mathcal{L}_2 \rangle P_{0,1} = -\langle \mathcal{M}_1 P_0 \rangle$$

This last expression and the terminal condition $P_{0,1}(T, x, r, z) = 0$ define completely $P_{0,1}$. $\langle \mathcal{L}_2 \rangle$ and $\langle \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle$ are defined by Bayraktar and Yang (2011) (4)¹⁵

¹⁴For a mathematical explanation see Fouque et. al. (2011) (22).

¹⁵Expressions (3.13) and (3.22).

Explicit Pricing Formula

Considering (11) for $i = j = 0$, $i = 1, j = 0$ and $i = 0, j = 1$, the first order approximation of $P^{\epsilon, \delta}$ is given by

$$\tilde{P}^{\epsilon, \delta} = P_0 + \sqrt{\epsilon}P_{1,0} + \sqrt{\delta}P_{0,1} \quad (21)$$

This is, the approximation is the sum of three terms: the leading term, the first order fast scale correction and the first order slow scale correction. These three terms do not depend on y and \tilde{y} (fast mean reverting factors) as we have seen in the last subsection so the size of the fluctuations of the derivatives' price does not change a lot. Additionally, this three terms have been completely well defined above by *solvability* and terminal conditions.

In order to compute the expectation in (5) we can use a change of probability measure. Therefore, with \mathbb{P}^T known as the T -forward measure (Fouque, et. al., 2011 (22)), we have to use a Radon–Nikodym derivative

$$\xi_T = \frac{d\mathbb{P}^T}{d\mathbb{P}} = \frac{\exp\left(-\int_0^T r_s ds\right)}{B(0, T)}$$

and the Radon–Nikodym derivative restricted to \mathcal{F}_t as defined in Fouque, et. al. (2011) (22)¹⁶ where $B(t, T)$ is the risk-free zero coupon bond defined above

$$\xi_t = \frac{d\mathbb{P}^T}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \mathbb{E}(\xi_T | \mathcal{F}_t) = \frac{\exp\left(-\int_0^t r_s ds\right) B(t, T)}{B(0, T)}$$

where ξ_t is the Radon–Nikodym process. Thereby, (5) turns to

$$\begin{aligned} P^{\epsilon, \delta}(t, X_t, r_t, Y_t, \tilde{Y}_t, Z_t) &= \mathbb{E} \left[\exp\left(-\int_t^T (r_s + l\lambda_s) ds\right) h(X_T) \Big| \mathcal{F}_t \right] = \\ &= \mathbb{E} \left[\exp\left(-\int_t^T r_s ds\right) \exp\left(-\int_t^T l\lambda_s ds\right) h(X_T) \Big| \mathcal{F}_t \right] = \\ &= \mathbb{E} \left[\exp\left(\int_0^t r_s ds\right) \exp\left(-\int_0^T r_s ds\right) \exp\left(-\int_t^T l\lambda_s ds\right) h(X_T) \Big| \mathcal{F}_t \right] = \\ &= \frac{B(t, T)}{B(0, T)} \mathbb{E} \left[\exp\left(\int_0^t r_s ds\right) \exp\left(-\int_0^T r_s ds\right) \frac{B(0, T)}{B(0, T)} \exp\left(-\int_t^T l\lambda_s ds\right) h(X_T) \Big| \mathcal{F}_t \right] \\ &= \frac{B(t, T)}{\xi_t} \mathbb{E} \left[\xi_T \exp\left(-\int_t^T l\lambda_s ds\right) h(X_T) \Big| \mathcal{F}_t \right] \end{aligned}$$

The last equality is because ξ_t is \mathcal{F}_t -measurable. Using the Lemma that establishes that for any adapted and integrable process C_t , $\mathbb{E}^T(C_t | \mathcal{F}_s) = \frac{1}{\xi_s} \mathbb{E}(\xi_t C_t | \mathcal{F}_s)$ ¹⁷ we finally get

$$\begin{aligned} P^{\epsilon, \delta}(t, X_t, r_t, Y_t, \tilde{Y}_t, Z_t) &= \frac{B(t, T)}{\xi_t} \mathbb{E} \left[\xi_T \exp\left(-\int_t^T l\lambda_s ds\right) h(X_T) \Big| \mathcal{F}_t \right] \\ &= B(t, T) \mathbb{E}^T \left[\exp\left(-\int_t^T l\lambda_s ds\right) h(X_T) \Big| \mathcal{F}_t \right] = B(t, T) F^{\epsilon, \delta}(t, X_t, r_t, Y_t, \tilde{Y}_t, Z_t) \end{aligned}$$

where $F^{\epsilon, \delta}$ being the price with the forward measure. Henceforth, Bayraktar and Yang (2011) (4) show that the deviations of $\tilde{P}^{\epsilon, \delta}$ from $P^{\epsilon, \delta}$ are bounded and small. Moreover, they develop closed-form expressions for each of the three terms of (21).

¹⁶Expression 12.106.

¹⁷Expression 1.59 in Fouque, et. al. (2011) (22).

HYPOTHESIS

Given the framework above, (21) is the proper price approximation and in order to get the best fitting, we only have to estimate the following parameters for European Call Options¹⁸

$$\{\bar{\lambda}(z), V_1^\epsilon(z), V_2^\epsilon, V_3^\epsilon(z), V_4^\epsilon, V_5^\epsilon, V_6^\epsilon, V_1^\delta(z), V_2^\delta(z)\}$$

Methodology

In this section we follow the methodology of calibration performed by Bayraktar and Yang (2011) (4). We perform cross-sectional estimations: estimations of the parameters while considering the different maturity times at a fixed moment in time (a fixed day). The parameters that will be estimated on a daily basis are given by the hypothesis. We are going to work with Ford Motor Company Stock and SPX Index European Call Options. Therefore, $l = 1$ and $h(X_T) = (X_T - K)^+$ in (5).

Data Description

- The daily closing stock price data comes from finance.yahoo.com.
- We use the U.S. government treasury yield data with maturities (number of observations)¹⁹: 1 month, 3months, 6 months, 1 year, 2 years, 3 years, 5 years, 7 years, 10 years, 20 years, 30 years.
- The stock option and corporate bond data are obtained from Bloomberg. Regarding the stock option data, Bayraktar and Yang (2011) (4) are able to use the OptionMetrics under WRDS database where the observed implied volatilities are already calculated. On the other hand, we have to apply the Black-Scholes formula to obtain the implied volatilities from the Bloomberg stock option data prices and we used the estimated parameter r , explained in the next subsection, to compute it. The number of available bond quotes and bond maturities vary and there are around 15 data points per day for bond data.
- $\bar{\rho}_1$ and $\bar{\sigma}_2$ are obtained as (i) the correlation between the 1-month treasury bonds as a proxy for the risk-free spot rate and the stock price, and (ii) as the estimated standard deviation from historical stock price data, respectively.

The Parameter Estimation

The parameters of the interest rate model (α, β, η, r) are obtained by non-linear least squares estimation of the Treasury Yield Curve of the corresponding day of analysis²⁰.

However, the U.S. government Treasury yield curve data has only 10 observations. To have more observations we use linear interpolation²¹, as in Chakroun and Abid, 2013 (11) getting 360 observations (monthly maturities) for the yield curve. Then following Rogers and Stummer (2000) (32), first we define for

¹⁸These are inside each term of (21) and are defined by propositions 3.1, 3.2 and 3.3 by Bayraktar and Yang (2011) (4). Each parameter is variable in a time dimension since they are defined by integrals over the fast mean reverting stochastic processes. However, within a day, they are constant.

¹⁹This data set is available at: <https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield>.

²⁰Van Elen (2010) (37) asserts that since there are 3 dimensions (yield, time to maturity and time), there is also a time series estimation method where the Vasicek model has to be discretized and then run an AR(1) model for the interest rate, rather than using the yield curve data, and maturity time is fixed.

²¹ $R(t) = R(t_i) + \left[\frac{(t-t_i)}{(t_{i+1}-t_i)} \right] \times [R(t_{i+1}) - R(t_i)]$ where i is the market observation index with time to maturity t_i and $R(t)$ corresponds to maturity t where $t_i \leq t \leq t_{i+1}$

day t and maturity T_j a vector $(R_t^{T_j})_{j=1}^M$ that will be the *yield to maturity*, where $M = 360$. Recall that $R_t^{T_j}$ is obtained by the affine structure of the Vasicek model. Thereby, the cross-section of the returns that will be estimated by non-linear least squares is²²

$$R_t^{T_j} = \left(\frac{\alpha}{\beta} - \frac{\eta^2}{2\beta^2} \right) + \left(r - \frac{\alpha}{\beta} + \frac{\eta^2}{\beta^2} \right) \frac{(1 - e^{-\beta(T_j-t)})}{\beta(T_j-t)} - \frac{\eta^2}{4\beta^3(T_j-t)} \left(1 - e^{-2\beta(T_j-t)} \right)$$

where $T_j - t$ is the only explanatory variable.

Step 1. Estimation of $\bar{\lambda}(z)$, $V_3^\epsilon(z)$ and $V_2^\delta(z)$ from the corporate bond price data

This step entails two sub-steps. The first sub-step consists in fixing a value of $\bar{\lambda}(z)$ and then estimating a $V_3^\epsilon(z)$ and $V_2^\delta(z)$ by linear least-squares estimation²³ with robust errors. The second substep consist in estimating $\bar{\lambda}(z)$ by non-linear least squares estimation²⁴ with robust errors subject to $\bar{\lambda}(z) \in [0, 1]$.

Step 2. Estimation of $V_1^\epsilon(z)$, V_2^ϵ , V_4^ϵ , V_5^ϵ , V_6^ϵ , $V_1^\delta(z)$ from the equity option data

While Bayraktar and Yang (2011) (4) work again with a cross-section estimation, in this step we perform a slightly different estimation due to our data set. As Chiarella et. al.(2007) (12) state, for each expiration date T_i , options with strike prices K_{ij} are traded. Then the cross-section estimation has two dimensions: different maturities and different strike prices. We perform again a non-linear least squares estimation²⁵ with robust errors on expression (4.6) in Bayraktar and Yang (2011) (4) since it gives us lower residual sum of squares.

Fitting the Implied Volatility of the Index Options

Since there is no default risk with Index Options because it does not depend on a particular defaultable company but on an equity index, $\bar{\lambda} = 0$. Then (21) becomes $\tilde{P}^{\epsilon,\delta} = P_0 + \sqrt{\epsilon}P_{1,0}$ and $\sqrt{\epsilon}P_{1,0}$ only will depend on 5 factors²⁶. Thus, we only have to perform the second step to obtain the estimated coefficients.

Results

Bayraktar and Yang (2011) (4) do not report the results of their estimations. They just show the value of the estimated coefficients, so we only are able to compare our estimated coefficients with theirs in terms of the magnitude.

The results of the yield curve estimation are reported in tables A.1. and A.2. We can see that the estimated coefficients are all significant and in figure A.1. it is shown that both estimated yield curves fit appropriately the observed interpolated yield curves²⁷. Yet, the estimated parameters are different from the ones reported by Bayraktar and Yang (2011) (4).

²² r should be r_t but since t is fixed, then $r_t = r$ is a constant here.

²³Where the left hand side is $B_{obs}^c(t, T_i) - B_0^c(t, T_i, \bar{\lambda}(z))$ and the right hand side is the addition of the two terms of expression (4.2) in Bayraktar and Yang (2011) (4).

²⁴Where the left hand side is $B_{obs}^c(t, T_i)$ and the right hand side is as the r.h.s. in expression (4.1) in Bayraktar and Yang (2011) (4) but plugging in the estimated coefficients $V_3^\epsilon(z)$ and $V_2^\delta(z)$.

²⁵Weighted as in expression (4.4) in Bayraktar and Yang (2011) (4).

²⁶ $g1$, $g2$, $g4$, $g5$, and $g6$, as defined by Bayraktar and Yang (2011) (4) in expression (4.5) and Appendix B.

²⁷The parameter r is a monthly rate so we turn it daily for the subsequent subsections.

For the Ford Motor Company Stock Call Options, the first step estimations in table A.3. shows very strongly significant coefficients, all of them positive. The price of credit risk $\lambda(z) = 0.0223$ is very close to Bayraktar and Yang's (2011) (4) (0.027)²⁸. However, considering also the second step significant coefficients in table A.4., we obtain different results from the authors' for the rest of parameters, much greater in absolute value²⁹.

With these parameters we generate the estimated implied volatility surface in figure A.3. for the April 26, 2017. We can see that it is similar to the observed one and it shows curvature but not as much as the observed and as Bayraktar and Yang's (2011) (4) since both volatility surfaces are more curved and smoother than the estimated in figure A.3. Also, in figure A.4. we can see that the goodness of fit is not as good as in Bayraktar and Yang (2011) (4) and is better the greater the maturity. Nevertheless, in their figure, these authors only show the points that have good fit, not all the points for all the strikes within each maturity.

For the SPX Index Call Options, the estimated coefficients in table A.5. are all strongly significant. Bayraktar and Yang (2011) (4) do not report the values of this coefficients. In addition, we can see on figures A.5. and A.6., in most of the maturities the estimated Implied Volatility Curve is on top of the observed one, or very close to it so the goodness of fit is better than for the Ford Motor Company. This could be because there are less parameters to estimate. Thus, the fit given by the Bayraktar and Yang's (2011) (4) model is satisfactory. However, the fit shown in their working paper with 2007 data is better than with our 2017 data.

Conclusions

The academic literature is still growing regarding the relationship between credit and equity derivatives where jump stochastic processes are crucial. Bayraktar and Yang (2011) (4) model is one of many models that wants to contribute to a unified framework able to explain this interaction.

In this Master Project we have explained all the mathematical theory beneath their model finding that there are strong fundamental bricks that support their theoretical model. We also have performed their methodology to compare their results with ours and we can reject the formulated hypothesis since the fitting is not as good as expected. This comparison brings us to Fouque et. al. (2011) (22) critic of this type of models: Bayraktar and Yang (2011) (4) model has a great fit with the data they used for two days, April 4 and June 8 of 2007. This type of modelling is endogenous to the financial industry viewpoint of a model's usefulness defined by simply mapping from prices to parameters. Nonetheless, there is a *trade-off* between goodness of fit and the stability of the parameters through time. Bayraktar and Yang (2011) (4) model goes just to one extreme because the financial industry may mistrust any model that does not fit a given day's data perfectly. In this way, our results show that the parameters are not stable through time, since we use newer data.

There are many ways to improve Bayraktar and Yang (2011) (4) model. As cited in the review of the literature, we can change the Vasicek model for more appropriate term structure models. We can also make the intensity endogenous³⁰. In addition, we can allow a positive equity recovery and we can adhere all this within an equilibrium model in order to assess macroeconomic implications of the interaction between credit an equity derivatives and their economic policy responses. Furthermore, the methodology has to be complemented with a time-series analysis to get a better equilibrium between goodness of fit and the stability of the parameters.

²⁸But this comes from the fact that we fix the value of this coefficient in the first sub-step and the value that we can fix it is an arbitrary value between 0 and 1.

²⁹ $(V_1^\epsilon(z), V_2^\epsilon, V_3^\epsilon(z), V_4^\epsilon, V_5^\epsilon, V_6^\epsilon, V_1^\delta(z), V_2^\delta(z)) = (0.9960, 0.0014, 0.0009, 0.0104, 0.6514, 0.3340, 0.1837, 0.0001)$

³⁰2005-2009 were bad years for Ford Motor Company but since then until know its average stock price (USD 13,63) is higher than the average on those years (USD 7,41). Thus, the intensity should change between good and bad times.

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Appendix

Ford Motor Company

The Ford Motor Company is a multinational firm dedicated in the manufacturing of commercial vehicles and luxury cars. As of today, Ford is a public listed company which is traded in the NYSE and is part of the S&P 500. Its weight is of 0.21% in the index and it has an equity capitalization of USD 4,083 billion and debt capitalization of USD 130 billion of which USD 93 billion are represented by bonds issued through capital markets.

Describe by Acharya, et. al. (2008) (1) and Fouque et. al. (2011) (22), in May 2005, Ford and GM companies were simultaneously downgraded to junk status and this caused a wide-spread sell-off in their corporate bonds. The reason was that many funds and banks were worried that their exposure to default risk was suddenly very high.

As a response, Ford implemented a plan to return the company to profitability. The Plan, named *The Way Forward* aimed to re-size the company to match market realities and to drop unprofitable and inefficient product lines: eventually Ford closed 14 factories and cut over 30,000 jobs. In December 2006, the company raised its debt capacity to about USD 25 billion: almost all corporate assets were necessary as collateral for this amount. In November 2007, Ford and the trade unions (United Auto Workers) agreed to a historic contract settlement, allowing the company to unload a lot of the economic burden given by health care costs and other retirement benefits. Ford Motor Company funded the creation of an independent company run by the Voluntary Employee Beneficiary Association (VEBA). The agreement was meant to improve Ford's balance sheet by shifting to this new company's balance sheet the burden of retiree and health care costs.

Ford would not have turned profitable until 2009. Through April 2009, Ford's strategy of debt for equity exchanges erased USD 9.9 billion in liabilities in order to leverage its cash position. For the first time after four years, Ford Motor Company posted a profit of USD 2,7 billion profit for 2009.

Table A.1.: 26-04-2017 Yield Curve Estimation for the Ford Motor Company Stock Call Options

Coefficients	Estimates	Std. Error	t-Statistic	p-value
α	0.0609	0.0079	7.72	0.000***
β	0.0135	0.0068	2.00	0.047**
η	0.0213	0.0002	2.98	0.003***
r	0.7917	0.0180	43.92	0.000***

Table A.2.: 06-01-2017 Yield Curve Estimation for the SPX Index Call Options

Coefficients	Estimates	Std. Error	t-Statistic	p-value
α	0.0961	0.0042	22.95	0.000***
β	0.0200	0.0039	5.11	0.000***
η	0.0362	0.0001	14.65	0.000***
r	0.4004	0.0152	26.36	0.000***

Figure A.1. Observed vs Estimated Yield Curves

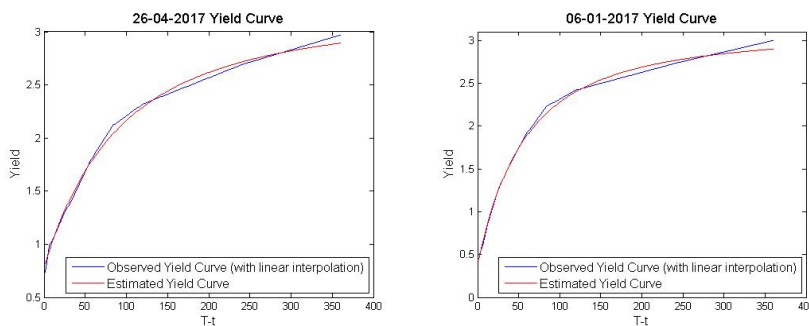


Table A.3.: First Step Estimation for the Ford Motor Company Stock Call Options

Coefficients	Estimates	Std. Error	t-Statistic	p-value
$\lambda(z)$	0.0223	0.0006	34.51	0.000***
$V_3^\epsilon(z)$	0.8712	0.1246	6.99	0.000***
$V_2^\delta(z)$	0.1901	0.0282	6.73	0.000***

Table A.4.: Second Step Estimated Coefficients for the Ford Motor Company Stock Call Options

Coefficients	Estimates	Std. Error	t-Statistic	p-value
$V_1^\epsilon(z)$	-0.1038	0.0344	-3.02	0.005**
$V_2^\epsilon(z)$	0.0404	0.0142	2.85	0.008**
$V_4^\epsilon(z)$	3.8464	0.6861	5.61	0.000***
$V_5^\epsilon(z)$	13.1405	2.4873	5.28	0.000***
$V_6^\epsilon(z)$	-23.8566	14.0140	-1.70	0.099*
$V_1^\delta(z)$	27.5202	14.5997	1.88	0.069*

In the next two figures we show the fit to the implied volatility of the Ford Motor Company on April 26, 2017 with maturities [23, 51, 142, 233, 268, 632] days. The model is calibrated across all maturities but here we show the implied volatility fits separately. The parameters are: stock price $x = 11.6$, $\bar{\sigma}_2 = 0.5$, $r = 0.095$, $\bar{\rho}_1 = -0.6$ and those in table 1. The black line with circles comes from the observed data whereas the red line with stars is generated by Bayraktar and Yang (2011) (4) Model.

Figure A.3.

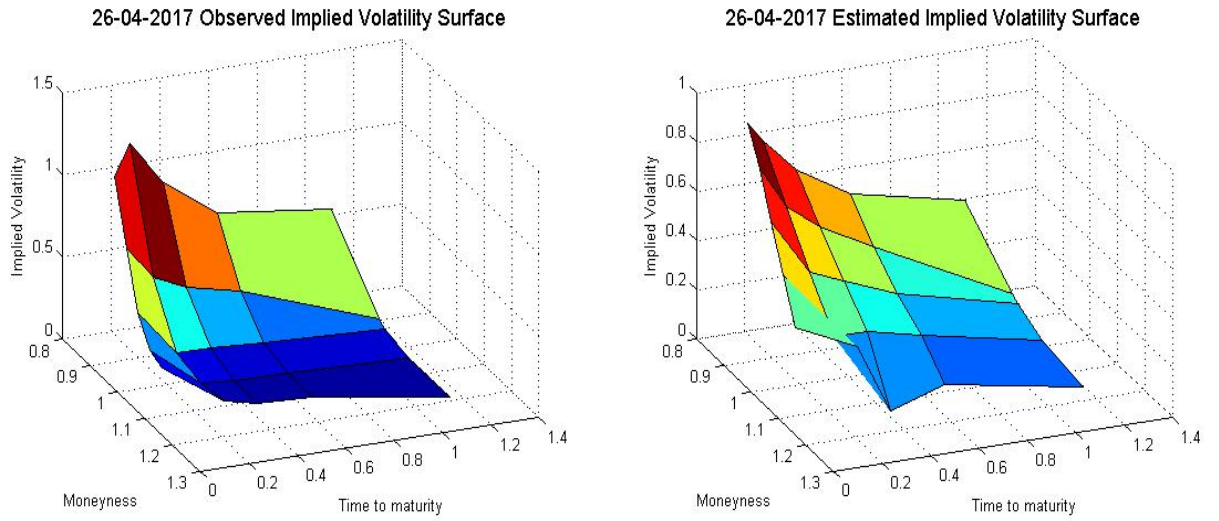
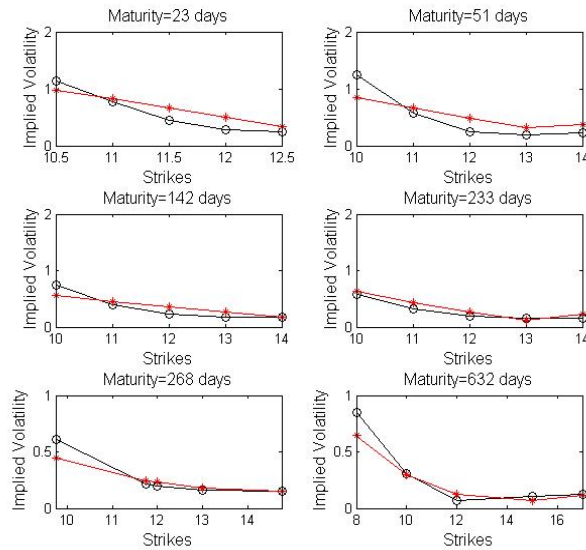


Figure A.4.



In the next two figures we show the fit to the implied volatility of SPX on January 6, 2017 with maturities [14, 25, 42, 53, 70, 84, 105, 112, 145, 161] days. The model is calibrated across all maturities but here we show the implied volatility fits separately. The parameters are: stock price $x = 2276.98$, $\bar{\sigma}_2 = 0.37$, $r = 0.048$, $\bar{\rho}_1 = -0.53$ and those in table 2. The black line with circles comes from the observed data whereas the red line with stars is generated by Bayraktar and Yang (2011) (4) Model.

Table A.5.: Estimated Coefficients for the SPX Index Call Options

Coefficients	Estimates	Std. Error	t-Statistic	p-value
$V_1^\epsilon(z)$	0.0870	0.0005	170.21	0.000***
$V_2^\epsilon(z)$	0.0138	0.0012	11.68	0.000***
$V_4^\epsilon(z)$	0.9700	0.0880	11.02	0.000***
$V_5^\epsilon(z)$	6.2370	0.5197	12.00	0.000***
$V_6^\epsilon(z)$	1.0475	0.0959	10.92	0.000***

Figure A.5.

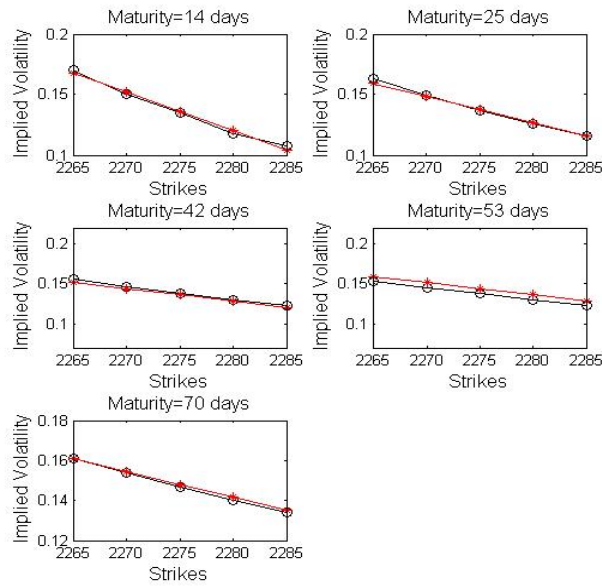


Figure A.6.

