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IMPLEMENTABILITY UNDER MONOTONIC TRANSFORMATIONS IN DIFFERENCES

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Abstract. Consider a social choice setting in which agents have quasilinear utilities over monetary transfers. A domain D of admissible valuation functions of an agent is called a revenue monotonicity domain if every 2-cycle monotone allocation rule is truthfully implementable (in dominant strategies) and satisfies revenue equivalence. We introduce the notions of *monotonic transformations in differences*, which can be interpreted as extensions of Maskin's monotonic transformations to quasilinear environments, and show that if D admits these transformations then it is a revenue monotonicity domain. Our proof is elementary and does not rely on strenuous additional machinery. We show that various economic domains, with *countable* or *uncountable* allocation sets, admit monotonic transformations in differences. Our applications include public and private supply of divisible public goods, multi-unit auction-like environments with increasing valuations, allocation problems with single-peaked valuations, and allocation problems with externalities.

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1. Introduction

In this paper, we consider allocation problems in a mechanism design setting in which agents have quasilinear utilities over monetary transfers and social alternatives. A valuation for an agent, which fully captures his relative preferences over alternatives, is a real function defined on the allocation set and is the agent's private information. We treat the set of admissible valuation functions —the *preference domain*— as our primitive. We study truthful (dominant strategy) implementability of direct revelation mechanisms, which are composed of an allocation rule mapping admissible valuations onto the allocation set, and an additional payment rule mapping profiles of valuations into monetary transfers. Our purpose is to contribute to the mechanism design program in its identification of settings where "well-behaved mechanisms" exist —i.e., mechanisms in which the allocation rule can be said to be truthfully implementable if it satisfies a system of constraints independent of payments, and for which any incentive compatible payment rule is expressible in term of the allocation rule alone.

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One of the earliest contributions comes from Myerson's (1981) auction design work, which shows that in single parameter settings one can replace the incentive constraints with a simple monotonicity requirement on the allocation rule. Once obtained, the monotone allocation rule could be used to completely recover the incentive compatible payments (up to a constant). Myerson's monotonicity condition does not extend easily to multi-dimensional settings. However it has been known since Rochet (1987) that a *cyclic monotonicity* condition on an allocation rule is equivalent to its truthful implementability in every quasilinear environment. In other words, Rochet showed that an allocation rule is truthfully implementable if, and only if, its corresponding allocation graph contains no cycle of negative length. Our contribution is finding a new set of sufficient attributes on the preference domain to ensure that the weaker *2-cycle monotonicity* is not only necessary but also sufficient for truthful implementability. What distinguishes our work from previous results available in the literature is that our conditions apply to finite *and* infinite allocation sets. As it turns out, these attributes also guarantee that every implementable allocation rule satisfies revenue equivalence. We illustrate the breadth of our results with some economic applications.

Section 2 contains the details of our model. For notational convenience, we deal only with one agent whose preference domain, denoted by D, is a subset of the space of real-valued functions defined on the allocation set A. We do not impose a priori restrictions on the cardinality of A in the derivation of our main result. D is called a revenue monotonicity domain if every 2-cycle monotone allocation rule is truthfully implementable and satisfies revenue equivalence. In Section 3 we introduce the notion of *monotonic transformations in differences*, which can be considered as adaptations to the quasilinear environment of Maskin's (1999) monotonic transformations so extensively employed in the social choice literature. Our main result, Theorem 1, states that if the D admits monotonic transformations in differences around one and two alternatives, then it is a revenue monotonicity domain. Our proofs, gathered in Section 4, are quite elementary. We believe that by getting rid of unnecessary assumptions, our results deepen our understanding of implementability and revenue equivalence. For instance, we derive an analogue of the Mirrlees representation of the indirect utility without relying on the envelope theorem formulation that requires a parametrization of preferences in terms of an auxiliary type space.

Our notion of monotonic transformations in differences rests on the possibility of distorting one or two valuations around one or two alternatives, respectively, in such a way that the value of each of the said alternatives is enhanced, relative to the value of other allocations, by the resulting transformations. Put differently, for valuation w in D and an alternative x in A, it should be possible to find a new valuation v in D such that the value difference between x and any other alternative a corresponding to v is strictly larger than the value difference between x and a corresponding to the original valuation w. In the left panel of Figure 1, this is accomplished by letting the value of x at v be equal to its value at w, and letting the value of all other alternatives at v be below the corresponding value at w. Monotonic transformations around two alternatives requires this type of distortions to occur, under certain circumstances, for pairs of valuations and alternatives. To help intuition, Section 3 contains simple examples of domains that admit monotonic transformations in differences but lie outside the scope of previous results in the literature.

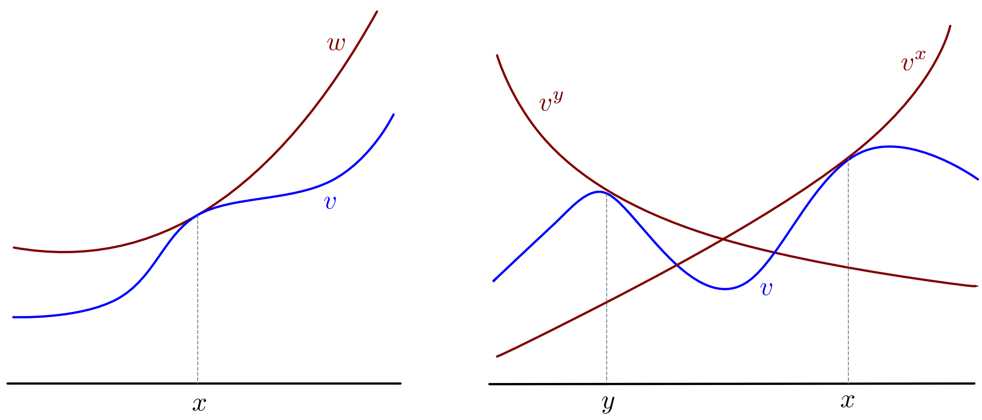

FIGURE 1. Transformations around one and two alternatives for $A = \mathbb{R}$

These conditions allow us to conclude that the sum of the lengths between any pair of alternatives in the allocation graph defined by a 2-cycle monotone allocation rule is exactly zero. From this we deduce several relevant consequences. First, if the allocation rule is implementable, then it must satisfy revenue equivalence. This is a direct outcome of the sum of the lengths between pairs of alternatives being equal to zero, a fact that is combined with the result in Heydenreich et al. (2009). Second, if an allocation rule is 3-cycle monotone then it must be cyclically monotone: indeed, one can transform the sum of the lengths in a 4-cycle to an equivalent sum of lengths of two adjacent 3-cycles, which have non-negative length. Applying this argument recursively yields to the desired conclusion. Third, given any 2-cycle monotone allocation rule, if there exists a 3-cycle with negative length in its corresponding allocation graph, then the reverse 3-cycle must have strictly positive length. The final step is to show that when D admits monotonic transformations in differences, no 2-cycle monotone allocation rule can generate a strictly positive 3-cycle.

Ashlagi et al. (2010) show that every 2-cycle monotone allocation rule that selects from a finite range of lotteries over finite alternatives is cyclically monotone.[1] They also show that if the closure of a domain (of dimension 2 or higher) is not convex, then there is a random finite-valued 2-cycle monotone allocation rule that is not implementable. Thus, while Ashlagi et al. (2010) provide a full characterization of 2-cycle monotonicity domains (which we do not), their effective allocation set is finite. To the best of our knowledge, our paper is the first to offer sufficient conditions for a revenue monotonicity domain that can be applied to both *finite non-convex* allocation sets with deterministic allocation rules and *infinite* allocation sets. Our interest in high cardinality alternative sets is undoubtedly theoretical but also stems from practical considerations. Indeed, in numerous applications the allocation set is modelled as a convex subset of an Euclidean space. Moreover, in

[1]For previous related sufficient conditions with finite allocation sets, see Saks and Yu (2005), Bikhchandani et al. (2006b), and Archer and Kleinberg (2008). Recently, Mishra et al. (2014) presents sufficient conditions based on ordinal properties.

some multi-dimensional design problems where revenue maximization is the objective, one cannot a priori assert the cardinality of the effective allocation set.[2]

In Section 6 we explore economic applications of our results. Section 6.1 deals with uncountable allocation sets and is motivated by the literature on public good provision (Green and Laffont (1977), Laffont and Maskin (1980), Güth and Hellwig (1986), among others), pollution rights (Dasgupta et al. (1980), Montero (2008)), network resource allocation (Kelly et al. (1998)), and quasilinear exchange economies (Goswami et al. (2014)). We prove that monotonic transformations in differences are admitted if A is an interval of the real line and the domain D consists of all continuous, non-negative, increasing functions. Monotonic transformations are also in place when the allocation set is a compact manifold and the domain of valuations is restricted to the set of all smooth functions. In all these cases, implementability and revenue equivalence are obtained from 2-cycle monotonicity. Unfortunately, the preference domain consisting of all concave valuations on a convex set does not satisfy monotonic transformations in differences around two alternatives.

The setting of Section 6.2 resembles those considered in multi-unit auction problems (e.g., Dobzinski and Nisan (2014)) or in other allocation problems with indivisible goods. Monotonic transformations are present when the allocation set is a countable ordered set and the domain of valuations consists of all increasing valuations on A. Single-peaked domains do not admit monotonic transformations around two alternatives. However, Mishra et al. (2014) have shown that single-peaked domains are revenue monotonicity domains. We pursue a pairwise version of monotonic transformations in differences that requires distortions around one alternative alone, and with this we are able to show that single-peaked domains and truncated domains are also revenue monotonicity domains.

In Section 6.3 we extend the allocation set to be the product space of two subsets of the real line (at least one of which is finite). We do so in order to model situations where externalities are present, as for instance models of technology licensing by an upstream monopolist to downstream competitors (Katz and Shapiro (1986)), or classic takeover models with atomistic stockholders (Grossman and Hart (1980), Burkart et al. (1998)), or models of diffusion of technology standards (Dybvig and Spatt (1983)). In all these cases, the valuation of the agent may be increasing in its first component (e.g., access to innovation) but decreasing in its second component (e.g., the number of competitors licensing the innovation). We show that here as well the domain of valuations allows for monotonic transformations in differences, thus every 2-cycle monotone allocation rule is implementable and satisfies revenue equivalence. Section 7 offers a few final remarks.

2. Preliminaries

The setting considered here is reminiscent of the models studied in Bikhchandani et al. (2006b), Chung and Olszewski (2007), Heydenreich et al. (2009), Ashlagi et al. (2010), among others. For simplicity of notation, we deal with single-agent environments. The results derived here remain unchanged in multi-agent environments when the solution concept is dominant strategy incentive compatibility.

[2] Manelli and Vincent (2007) present examples, in the context of a multi-good monopolist, where the optimal allocation rule entails randomization over outcomes and is not finite-valued.

There is a non-empty set A of alternatives, called the *allocation set*, which could be finite or infinite. Our agent has quasi-linear preferences over alternatives and monetary transfers; the utility he derives from $a \in A$ and payment $\rho \in \mathbb{R}$ is

$$v(a) \;-\; \rho.$$

The function $v\colon A \to \mathbb{R}$ is called a *valuation* and is private information. Let $D \subseteq \mathbb{R}^A$ denote the set of admissible valuations.[3] Throughout the paper, D is called the preference domain, or *domain* for short.

An *allocation rule* is a mapping

$$f\colon D \to A.$$

For each $a \in A$, denote by $f^{-1}(a) \subseteq D$ the subset of admissible valuations that select alternative a under f; i.e.,

$$f^{-1}(a) \;=\; \{v \in D \,:\; f(v) \,=\, a\}.$$

Except when explicitly mentioned, we restrict the analysis to surjective allocation rules, so $f^{-1}(a) \neq \emptyset$ for all $a \in A$. In our economic applications, we impose restrictions on A and D, with the understanding that properties ascribed to D may refer to both the allocation set A and the set of admissible valuations defined on A. For example, a domain D may be a subset of $\mathbb{R}^m$ when the allocation set has cardinality $|A| = m < \infty$, or the space of continuous or smooth functions defined on a metric (or topological) space, and so forth. A domain $D \subseteq \mathbb{R}^A$ coupled with an allocation rule f forms the *allocation problem* (D, f).

The designer observes D but not the actual realization of valuations. Thus, monetary transfers are in general required to induce truthful revelation. The allocation rule $f\colon D \to A$ is said to be *truthfully implementable* (in dominant strategies) if there is a payment rule $\pi\colon D \to A$ such that

$$v(f(v)) \;-\; \pi(v) \;\geq\; v(f(w)) \;-\; \pi(w), \qquad \text{for all } v, w \in D.$$

Let $\triangle v(a, b) \equiv v(a) - v(b)$ represent the value difference between a and b under v, for all $a, b \in A$, $v \in D$. Using this notation, f is implementable by π if and only if

$$\triangle v(f(v), f(w)) \;\geq\; \pi(v) \;-\; \pi(w), \qquad \text{for all } v, w \in D.$$

A *k-path in A* is a finite sequence of alternatives $\{a_1, a_2, \ldots, a_k\}$, where a_i belongs to A for each $i = 1, \ldots, k$. A *k-cycle in A* is a $(k+1)$-path in A with the additional provision that $a_{k+1} = a_1$. Say that $f\colon D \to A$ is *cyclically monotone* if for every integer $k \geq 2$, for every k-cycle $\{a_1, a_2, \ldots, a_k, a_{k+1} = a_1\}$ in A, and for all $v_i \in f^{-1}(a_i)$, $i = 1, \ldots, k$, one has

$$\sum\nolimits_{i=1}^{k} \triangle v_i(a_i, a_{i+1}) \;\geq\; 0. \tag{1}$$

Say that f is *2-cycle monotone* when Equation 1 is satisfied for all 2-cycles in A. Clearly, cyclic monotonicity always implies 2-cycle monotonicity. The converse of this observation is however not true in every allocation problem.[4]

Given (D, f), define for all pairs of alternatives $a, b \in A$, $a \neq b$,

$$\ell_f(a, b) \;\equiv\; \inf\,\{\triangle v(a, b) \,:\; v \in f^{-1}(a)\}. \tag{2}$$

[3]Here $\mathbb{R}^A$ denotes the set of all real-valued functions defined on A.

[4]A counter example is given in Section 4. See also Bikhchandani et al. (2006a) and Archer and Kleinberg (2008).

This expresses the minimal gains from truth-telling and getting alternative a instead of lying and obtaining b. Heydenreich et al. (2009) interpret this as the length, induced by f, of the direct arc connecting alternative a to b. Therefore, we refer to it as the *f-length from a to b*, and let $\ell_f(a,a) \equiv 0$ for all a in A. One can readily verify that f is cyclically monotone if, and only if, for every integer $k \geq 2$, every k-cycle $\{a_1, a_2, \ldots, a_k, a_{k+1} = a_1\}$ in A has non-negative f-length; i.e.,

$$\sum\nolimits_{i=1}^{k} \ell_f(a_i, a_{i+1}) \;\geq\; 0. \tag{3}$$

Similarly, f is 2-cycle monotone if, and only if, every 2-cycle $\{x, y, x\}$ in A has non-negative f-length; i.e.,

$$\ell_f(x,y) \,+\, \ell_f(y,x) \;\geq\; 0. \tag{4}$$

The relevance of cyclic monotonicity comes from the following result shown by Rochet (1987) —see also Rockafellar (1970).

Result 1. *For every domain D, the allocation rule $f\colon D \to A$ is truthfully implementable if, and only if, it is cyclically monotone.*

By the Taxation Principle, when f is implementable via π it must be that for all $a \in A$ and all valuations $v, w \in f^{-1}(a)$, $\pi(v) = \pi(w)$ —else the agent will find a profitable misreport. Thus, we associate the incentive compatible payment rule π with a real-valued function $p\colon A \to \mathbb{R}$ defined by

$$p(a) \;\equiv\; \pi(v), \qquad \text{for all } v \in f^{-1}(a), \text{ all } a \in A.$$

Without risk of confusion, we refer to p as a (non-linear) *price scheme* that implements f. Say that f satisfies *revenue equivalence* if for all price schemes $p, q\colon A \to \mathbb{R}$ that implement f, one has

$$p(a) \,-\, p(b) \;=\; q(a) \,-\, q(b), \qquad \text{for all } a, b \in A.$$

The *f-distance from a to b* is defined by

$$\operatorname{dist}_f(a,b) \;\equiv\; \inf\left\{ \sum\nolimits_{i=1}^{k-1} \ell_f(a_i, a_{i+1}) : \{a = a_1, \ldots, a_k = b\},\ k \geq 2 \right\},$$

where the infimum is taken over all paths in A connecting a to b. Heydenreich et al. (2009) showed the following characterization of revenue equivalence based on the f-distance —see also Kos and Messner (2013) and Rahman (2010).

Result 2. *In every allocation problem (D, f), the allocation rule f is truthfully implementable and satisfies revenue equivalence if, and only if,*

$$\operatorname{dist}_f(x,y) \,+\, \operatorname{dist}_f(y,x) \;=\; 0 \quad \textit{for all } x, y \in A.$$

3. Monotonic Transformations in Differences

D is called a *revenue monotonicity domain* if every 2-cycle monotone allocation rule $f\colon D \to A$ is truthfully implementable and satisfies revenue equivalence. Of course, not every domain is revenue monotone. Our main contribution is to establish sufficient attributes on D to be revenue monotone irrespective of the cardinality of A. These are captured by the notion of monotonic transformations in differences introduced in this section.

We begin with the following observation, which has been stated in different forms in various places before (it is included for completeness). Suppose that the price scheme p implements f. Then for all $x, y \in A$, all $v^x \in f^{-1}(x), v^y \in f^{-1}(y)$,

$$\triangle v^x(x,y) \;\geq\; p(x) \;-\; p(y) \;\geq\; \triangle v^y(x,y).$$

Combining this expression with Equation 2 obtains

$$\ell_f(x,y) \;\geq\; p(x) \;-\; p(y) \;\geq\; -\,\ell_f(y,x).$$

Immediately, a sufficient condition for revenue equivalence is that the f-length of any 2-cycle in A is exactly zero. A weaker sufficient condition is stated below.

Lemma 1. *Let $f\colon D \to A$ be truthfully implementable. If for all $x, y \in A$ there exists a finite path $\{x = a_1, a_2, \ldots, a_k = y\}$ such that*

$$\ell_f(a_i, a_{i+1}) \;+\; \ell_f(a_{i+1}, a_i) \;=\; 0 \qquad \textit{for all } i = 1, \ldots, k-1,$$

then f satisfies revenue equivalence.

Proof. If every consecutive 2-cycle $\{a_i, a_{i+1}, a_i\}$ in the path $\{x = a_1, a_2, \ldots, a_k = y\}$ connecting x to y has zero f-length, then one concludes that

$$0 \;=\; \sum\nolimits_{i=1}^{k-1} \ell_f(a_i, a_{i+1}) \;+\; \sum\nolimits_{i=1}^{k-1} \ell_f(a_{i+1}, a_i) \;\geq\; \mathrm{dist}_f(x,y) \;+\; \mathrm{dist}_f(y,x) \;\geq\; 0,$$

where the last inequality follows from the implementability of f. Applying Result 2 provides the desired conclusion. ☐

Suijs (1996) characterizes revenue equivalence for efficient allocation rules mapping into a finite allocation set in terms of Groves payments. One can verify that his result does not depend on any property of the allocation rule other than implementability. In particular, Suijs's (1996) Theorem 3.2 can be accommodated to show that when A is *finite*, the sufficient condition for revenue equivalence given in Lemma 1 is also necessary for all allocation rules, not just efficient ones. Necessity is however lost when the allocation set is infinite, as one may not be able to bound the number of nodes needed to connect pairs of alternatives by a sequence of tight 2-cycles —we illustrate this point with an example in Appendix B.

Intuition may suggest that the f-length of every 2-cycle in A adding up to zero would also serve as a sufficient condition to ascribe truthful implementability to f. This turns out to be *false*, even when the cardinality of A is finite (cf. Example 4). What suffices for truthful implementability is ensuring that both 2-cycles *and* 3-cycles have zero length. With this in mind, we introduce the notion of monotonic transformations for quasilinear preference domains. Henceforth, a given statement about an interval I of the real line is said to hold for *essentially all* δ in I if it holds for all but finitely many elements of I.

Definition 1. A domain D admits *bounded monotonic transformations in differences around one alternative (MD1)* if for all $x, y \in A$, $x \neq y$, for all $w \in D$ and all $\epsilon > 0$, there is a valuation $v \in D$ such that for every alternative $a \in A \setminus \{x\}$,

$$\triangle v(x,a) \;>\; \triangle w(x,a),$$

and the transformation v can be chosen to satisfy

$$\triangle v(x,y) \;<\; \triangle w(x,y) \;+\; \epsilon.$$

Under MD1 transformations, any valuation w in D can be distorted around a given alternative x to obtain a new admissible valuation that enhances the desirability of that alternative over everything else, relative to the original valuation. The role this property plays is analogous to the role Maskin's (1999) monotonic transformations play in the strategy-proofness literature[5]: if x is selected by a 2-cycle monotone allocation rule at w, and v is obtained from w using a MD1 transformation, then x is surely to be selected by the allocation rule at v as well. The presence of monetary transfers in quasi-linear environments requires stating these transformations in terms of valuation differences with respect to all alternatives. In addition, MD1 requires the penalty incurred by alternative $y \neq x$ under the transformed valuation to be arbitrarily small. We extend this idea to the two-alternative, two-valuation case.

Definition 2. A domain D admits *monotonic transformations in differences around two alternatives (MD2*)* if for all $x, y \in A$, $x \neq y$, all $v^x, v^y \in D$, and essentially all $\delta \in \mathbb{R}$ satisfying $\triangle v^x(x,y) > \delta > \triangle v^y(x,y)$, there is a valuation $v \in D$ such that $\triangle v(x,y) = \delta$ and for each alternative $a \in A \setminus \{x,y\}$, either

$$\triangle v(x,a) \;>\; \triangle v^x(x,a) \qquad \text{or} \qquad \triangle v(y,a) \;>\; \triangle v^y(y,a).$$

D admits *bounded monotonic transformations in differences around two alternatives (MD2)* if in addition for all distinct $x, y, z \in A$, all $v^x, v^y \in D$, all $\epsilon > 0$ and essentially all $\delta \in \mathbb{R}$ such that $\triangle v^x(x,y) > \delta > \triangle v^y(x,y)$, the above transformation v can be chosen to satisfy

$$\triangle v(x,z) \;<\; \triangle v^x(x,z) + \epsilon.$$

When $\triangle v^x(x,y) > \triangle v^y(x,y)$, under a MD2* transformation it is possible to find a transformation $v \in D$ that lowers the value difference between x to y with respect to the valuation v^x, thus making y more attractive relative to v^x, while at the same time increases the value difference between x and y with respect to v^y, thus making x more attractive relative to v^y. In addition, for any other alternative $a \in A \setminus \{x,y\}$, the transformed valuation v either enhances the value of x versus to a relative to v^x, or it enhances the value of y versus to a relative to v^y. Thus, when a 2-cycle monotone allocation rule selects x at v^x and y at v^y, and v in D is obtained from v^x and v^y using a MD2* transformation, then either x or y are to be selected by the allocation rule at v. Under MD2, in addition, it is possible to restrict the penalty incurred by alternative $z \neq x, y$ under v to an arbitrarily small amount relative to the values obtained at v^x.

Of course, if a domain admits MD2 transformations, then it also allows for the weaker MD2* transformations. Maintaining the distinction among the types of transformations will clarify the role each of them plays in our main contribution, which is the following theorem.

Theorem 1. *If D admits MD1 and MD2 transformations, then it is a revenue monotonicity domain.*

The proof of Theorem 1 is presented in Section 4. To help intuition, here we provide three examples of domains that admit MD1 and MD2 transformations but

[5]See for instance Barberà and Jackson (1995) and Serizawa and Weymark (2003). The recent paper of Goswami et al. (2014) explores quasilinear exchange economies with domains that do not admit Maskin monotonic transformations.

lie outside the models studied in Bikhchandani et al. (2006b), Ashlagi et al. (2010) or Mishra et al. (2014). The first example has a finite allocation set but a non-convex domain, and since one alternative is ranked as the preferred choice by every admissible valuation, it is not covered by Mishra et al. (2014). The last two have infinite allocation sets.

Example 1 (adapted from Vohra (2011)). There are two new, potentially complementary, production technologies to be adopted by a firm. The allocation set is $A = \{o, x, y, z\}$, where alternative o corresponds to the status quo, x and y denote the adoption of one of the novel production processes, and z is the acquisition of the bundle containing both technologies. A valuation (profit) function $w \in \mathbb{R}^A$ is admissible if:

$$\text{(i)} \quad 0 \;\leq\; w(o) \;<\; \min\{w(x), w(y)\}, \qquad \text{and}$$
$$\text{(ii)} \quad w(z) \;=\; \max\{w(x), w(y)\} \;+\; \kappa, \;\; \text{for some } 0 < \kappa < \overline{\kappa}.$$

The known parameter $\overline{\kappa}$ captures the *potential* degree of complementarities between the new technologies, but the *actual* degree is still private information of the firm. Let $D_I \subseteq \mathbb{R}^A$ contain all valuations that satisfy (i) and (ii). D_I is not convex, but admits MD1 and MD2 transformations, thus it is revenue monotone. ◇

Example 2. Consider a selling mechanism where the agent decides *when* to purchase a single indivisible object offered by a monopolist. The allocation set is a countable ordered set $A = \{a_1, a_2, \ldots, \}$, with each a_n representing the time period at which the agent receives the object and pays a price p_n. If our agent faces soft deadlines, the value assigned to the object is decreasing in time (albeit at different rates). The domain $D_{II} \subseteq \mathbb{R}^A$ is the set of all strictly decreasing, positive valuations defined on A. D_{II} allows for MD1 and MD2 transformations, hence is revenue monotone. ◇

Example 3. Let $A = [0, 1]$, as in various models of public good provision, or as in an exchange economy with quasilinear preferences where the total endowment of the perfectly divisible consumption good equals 1. If D_{III} is the space of all continuous functions defined on A, then it admits MD1 and MD2 transformations. ◇

Monotonic transformations are obtained in the first two examples using the next lemma. To simplify notation, for any $B \subseteq A$, let $\mathbb{1}_B$ denote the indicator function assigning value 1 to all $b \in B$ and 0 to all $a \in A \setminus B$. Given $w, \tilde{w} \in \mathbb{R}^A$, $w \vee \tilde{w}$ denotes the pointwise maximum; i.e.,

$$w \vee \tilde{w}(a) \;=\; \max\{w(a), \tilde{w}(a)\}, \qquad \text{for all } a \in A.$$

Say that D is *closed under positive shifts* if for all $w \in D, \xi > 0$, the valuation $w + \xi$ belongs to D. Say that D is *locally open* if for all $w \in D$, $B \subseteq A$ and $\xi > 0$ sufficiently small, the valuation $w + \xi \mathbb{1}_B$ lies in D.

Lemma 2. *If D is closed under positive shifts and locally open, and if $v \vee w \in D$ for all v, w in D, then D admits MD1 and MD2 transformations.*

Proof. We argue that (a stronger version of) MD2 transformations are admitted. Similar arguments show MD1. Fix distinct alternatives $x, y \in A$, $v^x, v^y \in D$, $\epsilon > 0$ and δ satisfying $\triangle v^x(x, y) > \delta > \triangle v^y(x, y)$. Choose $\xi^x, \xi^y > 0$ such that

$$\xi^x \;-\; \xi^y \;=\; \delta \;-\; v^x(x) \;+\; v^y(y).$$

Since D is closed under positive shifts, $\tilde{v}^x = v^x + \xi^x$ and $\tilde{v}^y = v^y + \xi^y$ belong to D. For such valuations, we have $\tilde{v}^x(x) - \tilde{v}^y(y) = \delta$, and

$$\begin{aligned} \tilde{v}^x(x) \;-\; \tilde{v}^y(x) \;&=\; \delta \;-\; \triangle v^y(x,y) \;>\; 0, \\ \tilde{v}^y(y) \;-\; \tilde{v}^x(y) \;&=\; \triangle v^x(x,y) \;-\; \delta \;>\; 0. \end{aligned}$$

Now define $v = \tilde{v}^x \vee \tilde{v}^y + \xi \mathbb{1}_{\{x,y\}}$, for $0 < \xi < \epsilon$ arbitrarily small. Since D is locally open and contains the pointwise maximum of any two valuations, $v \in D$. By construction, $\triangle v(x,y) = \delta$. Further, for $a \in A \setminus \{x,y\}$, if $v(a) = \tilde{v}^x(a)$ then

$$\triangle v(x,a) \;-\; \triangle v^x(x,a) \;=\; \triangle \tilde{v}^x(x,a) \;+\; \xi \;-\; \triangle v^x(x,a) \;=\; \xi.$$

On the other hand, if $v(a) = \tilde{v}^y(a)$ we conclude

$$\triangle v(y,a) \;-\; \triangle v^y(y,a) \;=\; \triangle \tilde{v}^y(y,a) \;+\; \xi \;-\; \triangle v^y(y,a) \;=\; \xi.$$

Note finally that in this last case, as $\tilde{v}^y(a) \geq \tilde{v}^x(a)$, one has

$$\begin{aligned} \triangle v(x,a) \;-\; \triangle v^x(x,a) \;&=\; \tilde{v}^x(x) \;+\; \xi \;-\; \tilde{v}^y(a) \;-\; v^x(x) \;+\; v^x(a) \\ &=\; \xi \;+\; \tilde{v}^x(a) \;-\; \tilde{v}^y(a) \;\leq\; \xi \;<\; \epsilon. \end{aligned}$$ □

The type of distortions in Lemma 2 cannot be applied to D_{III} without losing continuity. Despite this, it is possible to show that the space of continuously differentiable functions defined on $A = [0,1]$ admits MD1 and MD2 transformations, and we do so in Corollary 3. Here we only provide intuition for a key element applied to the continuous case. For each $x \in A$, for any subinterval B^x of A that contains x and is open (relative to A), there exists a continuous function μ^x mapping A to $[0,1]$ such that $\mu^x(x) = 1$, $0 < \mu^x(b) < 1$ for all $b \in B^x \setminus \{x\}$, and $\mu^x(a) = 0$ for all $a \in A \setminus B^x$. In particular, μ^x can be defined by the expression[6]

$$\mu^x(a) \;=\; 1 \;-\; \frac{|a-x|}{|a-x| \;+\; \inf\{|a-\tilde{a}| \colon \tilde{a} \in A \setminus B^x\}}.$$

Consequently, for any two alternatives $x, y \in A$ and open disjoint neighbourhoods $B^x, B^y \subset A$ containing x and y, respectively, we can combine μ^x and μ^y in an additive fashion to obtain a continuous function μ for which $\mu(x) = \mu(y) = 1$, $0 < \mu(b) < 1$ for all $b \in B^x \cup B^y$, and $\mu(a) = 0$ elsewhere.

4. Implementability from 2-Cycle Monotonicity

We divide the proof of Theorem 1 in a series of lemmas, emphasizing which type of transformation is employed in each particular instance. For the remaining of the section, fix an allocation problem (D, f).

Lemma 3. *If D admits MD2* transformations and f is 2-cycle monotone, then*

$$\ell_f(x,y) \;+\; \ell_f(y,x) \;=\; 0, \qquad \text{for all } x, y \in A.$$

Proof. To generate a contradiction, assume that $\ell_f(x,y) + \ell_f(y,x) > 0$ for some $x, y \in A$, $x \neq y$; the opposite inequality is ruled out by Eq. (4). Let $v^x \in f^{-1}(x)$ and $v^y \in f^{-1}(y)$, and choose $\delta \in \mathbb{R}$ to satisfy

$$\triangle v^x(x,y) \;\geq\; \ell_f(x,y) > \delta > -\ell_f(y,x) \;\geq\; \triangle v^y(x,y).$$

[6]Similar constructions have been used in the social choice literature before to prove, for instance, that the Gibbard-Satterthwaite Theorem holds on continuous domains over a metric space. See Barberà and Peleg (1990), and more recently Le Breton and Weymark (1999).

Using MD2*, we obtain $v \in D$ such that $\triangle v(x,y) = \delta$, and either

$$\triangle v(x,a) \;>\; \triangle v^x(x,a) \quad \text{or} \quad \triangle v(y,a) \;>\; \triangle v^y(y,a) \tag{5}$$

for all $a \in A \setminus \{x,y\}$. Combining Eq. (5) with the 2-cycle monotonicity of f, it follows that $v \in f^{-1}(x) \cup f^{-1}(y)$ and

$$\ell_f(x,y) \;>\; \triangle v(x,y) \;=\; \delta \;>\; -\,\ell_f(y,x).$$

But this is a contradiction —refer to Eq. (2). ☐

Suppose the f-length of every 2-cycle in A is exactly zero. When, in addition, every 3-cycle in A has non-negative length, it can be concluded that the length of any 4-cycle is also non-negative, as we can convert it into the sum of lengths of two adjacent 3-cycles —see Figure 2. Applied recursively, this logic yields to the following result.

Lemma 4. *Given (D,f), assume that $\ell_f(a,b) + \ell_f(b,a) = 0$ is satisfied for all alternatives $a,b \in A$. Then:*

(a) *For all $x,y,z \in A$, the 3-cycle $\{x,y,z,x\}$ has negative f-length if, and only if, the reverse 3-cycle $\{x,z,y,x\}$ has positive f-length.*
(b) *f is cyclically monotone if, and only if, $\ell_f(x,y) + \ell_f(y,z) + \ell_f(z,x) = 0$ for all $x,y,z \in A$.*

Proof. (a) Assume without loss of generality that x,y,z are distinct alternatives and suppose that

$$\ell_f(x,y) \;+\; \ell_f(y,z) \;+\; \ell_f(z,x) \;<\; 0.$$

Multiplying the above expression by minus one, rearranging, and using the fact that $-\ell_f(a,b) = \ell_f(b,a)$ for all $a,b \in A$, we obtain

$$\ell_f(x,z) \;+\; \ell_f(z,y) \;+\; \ell_f(y,x) \;>\; 0.$$

(b) Suppose that f is cyclically monotone. If the length of the 3-cycle $\{x,y,z,x\}$ is strictly positive, then by part (a) the length of the reverse cycle $\{x,z,y,x\}$ is negative, a contradiction.

Conversely, suppose the f-length of every 3-cycle is exactly zero. We argue by induction that the f-length of every k-cycle $\{a_1, a_2, \ldots, a_k, a_{k+1} = a_1\}$ in A is non-negative. By assumption, for any $x,y,z \in A$, we have $\ell_f(x,y) + \ell_f(y,x) = 0$ and further $\ell_f(x,y) + \ell_f(y,z) + \ell_f(z,x) = 0$. Suppose that Eq. (3) holds for all $3 \le k \le K-1$. We show that it remains true for K. Let $\{a_1, a_2, \ldots, a_K, a_{K+1} = a_1\}$ be an arbitrary K-cycle in A, and write

$$\begin{aligned}
\sum\nolimits_{i=1}^{K} \ell_f(a_i, a_{i+1}) \;&=\; \ell_f(a_1,a_2) + \ldots + \ell_f(a_{K-2},a_{K-1}) + \ell_f(a_{K-1},a_K) + \ell_f(a_K,a_1) \\
&=\; \ell_f(a_1,a_2) + \ldots + \ell_f(a_{K-2},a_{K-1}) + \ell_f(a_{K-1},a_1) \\
&\qquad + \ell_f(a_1,a_{K-1}) + \ell_f(a_{K-1},a_K) + \ell_f(a_K,a_1) \\
&\ge\; 0,
\end{aligned}$$

where the second equality follows from the fact that $\ell_f(a_{K-1},a_1) + \ell_f(a_1,a_{K-1}) = 0$, and the third follows from our induction step. ☐

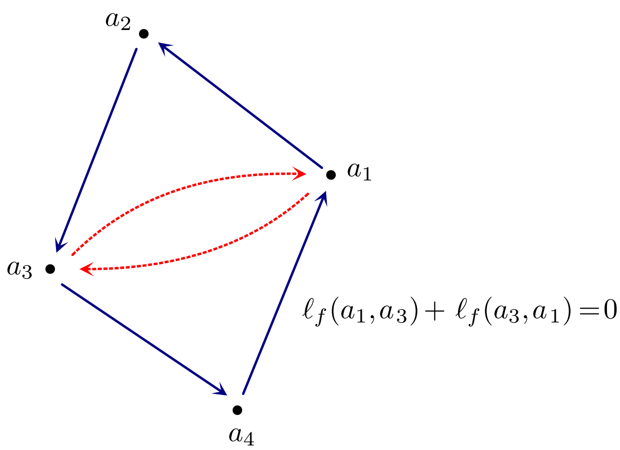

FIGURE 2. 4-cycle monotonicity from 3-cycle monotonicity

An important consequence of Lemma 3 together with Lemma 4 is that, under MD2* transformations, verifying that the length of every 3-cycle induced by a 2-cycle monotone allocation rule is zero suffices for truthful implementability. But MD2* does not guarantee zero length 3-cycles, not even in the finite allocation case.

Example 4. The set of alternatives is $A = \{x, y, z\}$. A valuation is represented by a vector $v = (v(x), v(y), v(z)) \in \mathbb{R}^3$. D_α, D_β and D_γ are subsets of $\mathbb{R}^3$ defined by

$$\begin{aligned}
D_\alpha &= \big\{(14+\alpha_x,\, 8,\, 2+\alpha_z) : 0<\alpha_x<1,\, \alpha_x<\alpha_z<1+\alpha_x\big\},\\
D_\beta &= \big\{(15+\beta_x, 10+\beta_y, 6) : 0<\beta_y<1,\, \beta_y<\beta_x<1+\beta_y\big\},\\
D_\gamma &= \big\{(10,\, 2+\gamma_y,\, \gamma_z-1) : 0<\gamma_z<1,\, \gamma_z<\gamma_y<1+\gamma_z\big\}.
\end{aligned}$$

The domain D_{IV} of admissible valuations is equal to $D_\alpha \cup D_\beta \cup D_\gamma$. We show in Appendix B that D_{IV} admits MD2* transformations but misses MD2.

Let f be an allocation rule that chooses x on D_α, y on D_β, and z on D_γ. Readily

$$\begin{aligned}
\ell_f(x,y) &= 6 = -\ell_f(y,x),\\
\ell_f(y,z) &= 4 = -\ell_f(z,y),\\
\ell_f(z,x) &= -11 = -\ell_f(x,z).
\end{aligned}$$

The f-length of every 2-cycle in the allocation graph of (D_{IV}, f) is zero, but the 3-cycle $\{x, y, z, x\}$ has negative length, therefore f fails to be implementable. $\diamond$

To obtain our main result, we rely on MD1–MD2 transformations to ensure that every 3-cycle in the allocation set has zero length under 2-cycle monotonicity. The intuition behind this last step can be formulated along these lines. Suppose that f is 2-cycle monotone and the domain D admits MD1 and MD2 transformations. Using MD1, we find v^x and v^y in D such that f chooses x at v^x and y at v^y. Moreover, these valuations can be obtained in a way that makes the value differences $\triangle v^x(x,z)$ and $\triangle v^y(y,z)$ arbitrarily close to $\ell_f(x,z)$ and $\ell_f(y,z)$, respectively. Using MD2, it is possible to find an admissible valuation v for which (i) the value difference $\triangle v(x,y)$ is arbitrarily close to $\ell_f(x,y)$; (ii) the value differences $\triangle v(x,z)$ and $\triangle v(y,z)$ are arbitrarily close to $\triangle v^x(x,z)$ and $\triangle v^y(y,z)$, and (iii) f selects alternative x or alternative y at v. Therefore, we deduce that $\ell_f(x,y)$ is arbitrarily close to $\ell_f(x,z)+\ell_f(z,y)$. The next two lemmas make this intuition rigorous.

Lemma 5. *If D satisfies MD1 and f is 2-cycle monotone, then for all distinct allocations $x,y,z \in A$, for all $\epsilon > 0$, there exist valuations $v^x \in f^{-1}(x), v^y \in f^{-1}(y)$ that satisfy the relations:*

$$\ell_f(x,z) \;<\; \triangle v^x(x,z) \;<\; \ell_f(x,z) + \epsilon, \tag{6}$$

$$\ell_f(y,z) \;<\; \triangle v^y(y,z) \;<\; \ell_f(y,z) + \epsilon, \tag{7}$$

$$\triangle v^y(x,y) \;<\; -\ell_f(y,x) \;\le\; \ell_f(x,y) \;<\; \triangle v^x(x,y). \tag{8}$$

Proof. Fix distinct $x,y,z \in A$ and $\epsilon > 0$. Clearly, one can find $w^x \in f^{-1}(x)$ and $w^y \in f^{-1}(y)$ for which the following relations are satisfied:

$$\begin{aligned} \ell_f(x,z) \;&\le\; \triangle w^x(x,z) \;<\; \ell_f(x,z) + \epsilon, \\ \ell_f(y,z) \;&\le\; \triangle w^y(y,z) \;<\; \ell_f(y,z) + \epsilon, \\ \triangle w^y(x,y) \;\le\; -\ell_f(y,x) \;&\le\; \ell_f(x,y) \;\le\; \triangle w^x(x,y) \end{aligned}$$

Let $\epsilon^x = \ell_f(x,z) + \epsilon - \triangle w^x(x,z) > 0$. Using a MD1 transformation obtains a valuation $v^x \in D$ such that $\triangle v^x(x,a) > \triangle w^x(x,a)$ for all $a \neq x$. Further, v^x can be chosen to bound the penalty incurred by z to $\triangle v^x(x,z) < \triangle w^x(x,z) + \epsilon^x = \ell_f(x,z) + \epsilon$. Since f is 2-cycle monotone, this shows that $v^x \in f^{-1}(x)$ satisfies Eq. (6) and the last inequality in Eq. (8). A similar argument applies to finding a corresponding $v^y \in f^{-1}(y)$ for which Eq. (7) and the first inequality in Eq. (8) are satisfied as well. ☐

Lemma 6. *If D satisfies MD1–MD2 and f is 2-cycle monotone, then*

$$\ell_f(x,y) + \ell_f(y,z) + \ell_f(z,x) \;=\; 0, \quad \textit{for all } x,y,z \in A.$$

Proof. Let $x,y,z \in A$ be given. When $x = y = z$ the result is trivial, and when only two of the three given alternatives are different it follows from Lemma 3. Thus, assume x,y,z are all distinct and fix an arbitrary $\epsilon > 0$. Lemma 5 furnishes $v^x \in f^{-1}(x)$ and $v^y \in f^{-1}(y)$ such that Equations (6), (7) and (8) are satisfied, with $\ell_f(x,y) = -\ell_f(y,x)$ in Eq. (8) because MD2* is present. Now choose $0 < \epsilon' < \epsilon$ such that

$$\triangle v^y(x,y) \;<\; \ell_f(x,y) - \epsilon' \;<\; \triangle v^x(x,y).$$

For $\delta = \ell_f(x,y) - \epsilon'$, using a MD2 transformation yields $v \in D$ such that $\triangle v(x,y) = \delta$ and for every $a \in A$, $a \neq x,y$, either $\triangle v(x,a) > \triangle v^x(x,a)$ or $\triangle v(y,a) > \triangle v^y(y,a)$ holds. The 2-cycle monotonicity of f implies that $f(v) \in \{x,y\}$. Moreover, since $\triangle v(x,y) = \delta < \ell_f(x,y)$, we conclude $f(v) = y$. In addition, v can be chosen to satisfy $\triangle v(x,z) < \triangle v^x(x,z) + \epsilon'$. Therefore,

$$\ell_f(y,z) \;\le\; \triangle v(y,z) \;=\; \triangle v(y,x) + \triangle v(x,z) \;<\; -\ell_f(x,y) + \triangle v^x(x,z) + 2\epsilon'.$$

We combine Eq. (6) with this last expression to obtain

$$\ell_f(x,y) + \ell_f(y,z) \;<\; \ell_f(x,z) + \epsilon' + 2\epsilon.$$

As $-\ell_f(x,z) = \ell_f(z,x)$, and because the preceding argument is valid for arbitrarily small $0 < \epsilon' < \epsilon$, this yields to

$$\ell_f(x,y) + \ell_f(y,z) + \ell_f(z,x) \;\le\; 0.$$

Exchanging the roles of y and z, we deduce that the f-length of the cycles $\{x, y, z, x\}$ and $\{x, z, y, x\}$ are both non-positive. Now suppose that $\ell_f(x, y) + \ell_f(y, z) + \ell_f(z, x) < 0$. Then, using Lemma 4(1), we conclude that $\ell_f(x, z) + \ell_f(z, y) + \ell_f(y, x) > 0$, which contradicts our previous findings. It follows that

$$\ell_f(x, y) + \ell_f(y, z) + \ell_f(z, x) = 0. \qquad \square$$

Proof of Theorem 1. Combining Lemmas 3, 4, 5, and 6 obtains cyclic monotonicity from 2-cycle monotonicity. Lemma 1 provides revenue equivalence. □

An interesting observation is that our transformations provide sufficient richness to the domain to permit the reformulation of an allocation problem —i.e., finding an allocation rule $f : D \to \mathbb{R}$ and corresponding incentive payment rule $\pi : D \to \mathbb{R}$— as one where the objective is to find a well-defined system of prices to satisfy

$$p(a) - p(a_0) = \ell_f(a, a_0) \equiv \inf \{v(a) - v(a_0) : v \in f^{-1}(a)\} \tag{9}$$

for a 2-cycle monotone allocation rule (fixing arbitrarily an alternative $a_0 \in A$). In this dual formulation, Equation 9 plays the role of the Mirrlees representation of the price schemes derived under various assumptions in parameterized mechanism design environments (e.g., convexity or differentiability of the valuation with respect to types). This representation, we believe, will prove to be useful in applied design problems —for instance when the objective of the designer can be stated in terms of revenue maximization— as it facilitates writing the optimization problem as a saddle-point problem.[7]

5. Extensions

5.1. Contractions

Given domain D and surjective allocation rule $f : D \to A$, let $f|_{D'}$ denote the restriction of f to $D' \subseteq D$. The allocation problem (C, g) is said to be a *contraction of* (D, f) if it is the case that $C \subseteq D$ and $g : C \to A$ is a surjective allocation rule satisfying $g = f|_C$. If f satisfies 2-cycle monotonicity, then it follows readily that g will also be 2-cycle monotone. This is because when (C, g) is a contraction of (D, f), for all $x, y \in A$ one gets from the definition of the length between two alternatives that

$$\ell_g(x, y) = \inf \{\triangle v(x, y) : v \in g^{-1}(x)\} \geq \inf \{\triangle v(x, y) : v \in f^{-1}(x)\} = \ell_f(x, y).$$

This implies that the f-lengths of cycles in the allocation graph of g are larger than or equal to the f-lengths of cycles with the same nodes in the graph of f.

The next observation plays a role in a few applications of Section 6.

Lemma 7. *Let (D, f) be a given allocation problem and (C, g) a contraction of (D, f). If f is 2-cycle monotone and C admits MD2* transformations, then*

$$\ell_g(x, y) = \ell_f(x, y), \qquad \textit{for all } x, y \in A.$$

[7]We would like to thank Jean-Charles Rochet for this observation.

Proof. By the argument put down prior to the lemma, we know that g is 2-cycle monotone. If C allows for MD2*, from Lemma 3 we conclude that $\ell_g(x,y) + \ell_g(y,x) = 0$ for all $x, y \in A$. Henceforth, the relations

$$-\ell_g(y,x) \;\leq\; -\ell_f(y,x) \;\leq\; \ell_f(x,y) \;\leq\; \ell_g(x,y)$$

imply that $\ell_f(x,y) = \ell_g(x,y)$. □

5.2. Pairwise Monotonic Transformations

MD2* and MD2 transformations require distortions of valuations around two alternatives, leaving single-peaked domains outside their scope.[8] However, Mishra et al. (2014) have shown that (generalized) single-peaked domains on a finite allocation set are revenue monotonicity domains. Inspired by their work, we introduce the related notion of *pairwise monotonic transformations in differences*. For the remainder of this section, let $A = \{a_1, a_2, \ldots\}$ be a countable set order-isomorphic to $\mathbb{N}$, so that $a_m \prec a_n$ if and only if $m < n$. When alternative a is the immediate predecessor or the immediate successor of alternative b, we refer to them as a *pair of consecutive elements* of A and write $\langle a, b\rangle$.[9]

Definition 3. A domain D admits *pairwise monotonic transformations in differences (PMD*)* if for all consecutive pairs $\langle x, y\rangle$ in A, all $w \in D$ for which $\triangle w(x,y) \geq 0$, all $\epsilon > 0$ and essentially all $\delta \in \mathbb{R}$ satisfying $\triangle w(x,y) > \delta > -\epsilon$, there exists $v \in D$ such that $\triangle v(x,y) = \delta$ and for every alternative $a \neq x, y$,

$$\triangle w(x,a) \;<\; \triangle v(x,a).$$

D admits *bounded pairwise monotonic transformations in differences (PMD)* if in addition for all $x, y, z \in A$ such that $\langle x, y\rangle$ and either $z \prec y \prec x$ or $x \prec y \prec z$, for all $w \in D$ for which $\triangle w(x,y) \geq 0$, all $\epsilon > 0$ and essentially all $\delta \in \mathbb{R}$ such that $\triangle w(x,y) > \delta > -\epsilon$, the transformation v can be chosen to satisfy

$$\triangle v(x,z) \;<\; \triangle w(x,z) + \epsilon.$$

Under a PMD* transformation, for arbitrary $\langle x, y\rangle \in A$ and $w \in D$, it is possible to find an admissible valuation v that increases the value of x relative to any other alternative a at w, with the exception of alternative y whose value increases relative to x. PMD requires in addition that the penalty applied to alternative z be arbitrarily small, but z is restricted to be a successor of y when y is a successor of x, or a predecessor of y when y itself is a predecessor of alternative x.

Theorem 2. *If D admits MD1 and PMD transformations, then it is a revenue monotonicity domain.*

We argue the validity of this theorem in a manner that mimics the arguments of Theorem 1, relegating the proofs to Appendix A.

Lemma 8. *If D admits PMD* and f is 2-cycle monotone, then $\ell_f(x,y) + \ell_f(y,x) = 0$ for all consecutive pairs $\langle x, y\rangle$ in A.*

[8] Let $A = \{x, y, z\}$ be such that $z \prec y \prec x$. Consider valuations v^x, v^z with peaks at x and z, respectively. Then $\triangle v^x(x,z) > 0 > \triangle v^z(x,z)$. For $\delta = 0$, any v that satisfies $\triangle v(x,z) = 0$ and decreases the value of y relative to either v^x or v^z violates single-peakedness.

[9] At some notational cost, one could alternatively assume that A is countable and a priori endowed with a family E of two-alternative subsets of A, called edges and represented by $e = \{a, b\}$, with the graph $G = (A, E)$ being connected.

The above lemma is weaker than Lemma 3 as it only ensures that the 2-cycles formed by consecutive pairs have zero length. Because of that, we also derive a weaker analogue to Lemma 4.

Lemma 9. *Given (D, f), assume that $\ell_f(a,b) + \ell_f(b,a) = 0$ is satisfied for all consecutive pairs $\langle a,b\rangle$ in A. If in addition the inequality*

$$\ell_f(x,z) \;\geq\; \ell_f(x,y) \,+\, \ell_f(y,z)$$

holds for all consecutive pairs $\langle x,y\rangle$ and all $z \in A$ such that either $z \prec y \prec x$ or $x \prec y \prec z$, then f is cyclically monotone.

The last step is given by the following result.

Lemma 10. *If D admits MD1 and PMD and if f is 2-cycle monotone, then*

$$\ell_f(x,z) \;\geq\; \ell_f(x,y) \,+\, \ell_f(y,z),$$

for all consecutive pairs $\langle x,y\rangle$, for all $z \in A$ such that either $z \prec y \prec x$ or $x \prec y \prec z$.

As before, a Mirrlees representation of the incentive compatible price scheme can be obtained based on the f-length. In particular, letting $A = \{a_0, a_1, a_2, \ldots,\}$ be fully ordered by $a_n \prec a_m$ if and only if $n < m$, a marginal pricing rule emerges. Under MD1 and PMD, the price scheme $p\colon A \to \mathbb{R}$ truthfully implements a 2-cycle monotone allocation rule f if and only if for every positive integer k,

$$p(a_k) \,-\, p(a_0) \;=\; \sum\nolimits_{i=1}^{k} \ell_f(a_i, a_{i-1}) \;=\; p(a_{k-1}) \,+\, \ell_f(a_k, a_{k-1}).$$

6. Applications

We now describe different economic applications of our main theorems on revenue monotonicity domains. The proofs of results are somewhat tedious and do not provide additional insights, thus we gather them in Appendix A.

6.1. Implementability on Large Allocation Sets

The problem of public good provision under quasilinear preferences and asymmetric information has received much attention in the mechanism design literature, with a substantial part of the work focusing on efficiency —public provision, e.g., Green and Laffont (1977), Laffont and Maskin (1980), Ledyard and Palfrey (1999)— and, to a lesser extend, on revenue maximization —private provision, e.g., Güth and Hellwig (1986) and more recently Csapó and Müller (2013). Most of this work assumes that the public good is perfectly divisible, so that the allocation set is $A = [0,1]$, or some other convex subset of the real line or an Euclidean space. A similar structure on the allocation set is assumed in models of pollution rights allocation —e.g., Dasgupta et al. (1980), Montero (2008); network resource allocation —e.g., Kelly et al. (1998) Johari and Tsitsiklis (2004); and exchange economies with quasilinear preferences —e.g., Goswami et al. (2014). It has been known for some time that if the admissible domain of valuations is the space of continuous (or the space of differentiable functions) on a metric space of allocations, then efficient allocation rules can be implemented uniquely using VCG transfers —thus revenue equivalence holds for efficient allocation rules, c.f. Green and Laffont (1977). Our results demonstrate that under different configurations on the allocation environment, revenue equivalence holds for *all implementable allocation rules.* Moreover, one can replace the incentive compatibility conditions with 2-cycle monotonicity even when the allocation set has the cardinality of the continuum.

Corollary 1. *Let $A = [\underline{a}, \overline{a}]$ be a compact interval of the real line and $D_1 \subseteq \mathbb{R}^A$ be the space of all non-negative, continuous, strictly increasing functions on A. Then every 2-cycle monotone allocation rule $f \colon D_1 \to A$ is truthfully implementable and satisfies revenue equivalence.*

The proof of Corollary 1 relies heavily on the fact that the domain of valuations admits only strictly increasing functions. The next corollary shows that this extends to the case of weakly increasing functions as well, provided the allocation problem at hand admits an appropriate contraction.

Corollary 2. *Let $A = [\underline{a}, \overline{a}]$ be a compact interval of the real line and $D_2 \subseteq \mathbb{R}^A$ be the space of all non-negative, continuous, weakly increasing functions on A. Let $f \colon D_2 \to A$ be surjective and 2-cycle monotone and suppose that (D_2, f) admits a contraction on $D_1 \subset D_2$. Then f is truthfully implementable and satisfies revenue equivalence.*

The next corollary extends the setting of Example 3.

Corollary 3. *For any $1 \leq r < \infty$, let A be a compact $\mathscr{C}^r$ manifold and $D_3 \subseteq \mathbb{R}^A$ be the space of all $\mathscr{C}^r$ real-valued functions on A. Then every 2-cycle monotone allocation rule $f \colon D_3 \to A$ is truthfully implementable and satisfies revenue equivalence.*

6.2. Implementation on Small Allocation Sets

From Example 2 we conclude that if $A = \{a_1, a_2, \dots, \}$ is a countable set ordered by the complete binary relation $a_m \succ a_n$ if and only if $m > n$, and if D is the set of all strictly increasing, non negative valuations defined on A, then D is a revenue monotonicity domain. Using Lemma 7, we obtain the following result.

Corollary 4. *Let $A = \{a_1, a_2, \dots\}$ be a countable ordered set such that $a_m \succ a_n$ if and only if $m > n$, and $D_4 \subseteq \mathbb{R}^A$ the space of all non-negative, weakly increasing real-valued functions on A. Let $f \colon D_4 \to A$ be 2-cycle monotone and suppose that (D_4, f) admits a contraction on $D \subset D_4$, where D is the set of all non-negative, strictly increasing functions on A. Then f is truthfully implementable and satisfies revenue equivalence.*

Corollary 4 can be applied to multi-unit auction environments, as in Dobzinski and Nisan (2014), where the valuation of an agent is increasing in the number of units she receives. It uses the fact that the domain of strictly increasing valuations on $A = \{a_1, a_2, \dots\}$ admits MD2 distortions. But single-peaked preference domains do not, irrespective of the cardinality of the allocation set. Thus we rely on pairwise monotonic transformations in differences to obtain truthful implementability and revenue equivalence in this case.

Corollary 5. *Let $A = \{a_1, a_2, \dots\}$ be a countable ordered set such that $a_m \succ a_n$ if and only if $m > n$. Let $D_5 \subseteq \mathbb{R}^A$ be the space of all single-peaked valuations defined on A. If $f \colon D_5 \to A$ is 2-cycle monotone, then it is truthfully implementable and satisfies revenue equivalence.*

The above corollary does not include models of single-peak preferences on uncountable allocation sets that have been used in the social choice literature to study allotment of divisible tasks among various participants —e.g., assigning assets to different creditors on a bankrupt procedure, as in Barberà et al. (1997). However, as our Corollary 5 covers single-peaked valuations on a countable ordered set, we could study approximate versions of the aforementioned models in settings that permit monetary transfers among participants. Additional applications of PMD include truncated preference domains where agents' report values for a strict subset of alternatives. This type of preferences has been used in the market design literature —e.g., Abdulkadiroğlu et al. (2005)— to model situations where participants are asked to submit rankings of few, among the many, possible alternatives. This accommodates environments where agents are allowed to report valuations on "nearby" alternatives —e.g., in a college dorm allocation problem an agent reports a few options located in the same building or in the same neighborhood. Say that $w\colon A \to \mathbb{R}$ is a *truncated valuation* if there exist a finite sequence of consecutive elements $\langle b_1, \ldots, b_k\rangle$, and a constant $\kappa \in \mathbb{R}$ such that $w(a) = \kappa < \max\{w(b_1), \ldots, w(b_k)\}$, for all $a \in A \setminus \langle b_1, \ldots, b_k\rangle$.

Corollary 6. *Let A be a countable ordered set and $D_6 \subseteq \mathbb{R}^A$ be the space of all truncated valuations defined on A. If $f\colon D_6 \to A$ is 2-cycle monotone, then it is truthfully implementable and satisfies revenue equivalence.*

6.3. Implementation on Mixed Allocation Sets

In some multi-agent mechanism design environments, an alternative a is modelled as a tuple $(a_1, \ldots, a_n)$ belonging to a subset of a n-dimensional Euclidean space, so that $a_i \geq 0$ may be representing the amount of a given resource assigned to agent $i = 1, \ldots, n$. For example, a_i may be the quantity of an intermediate good received by firm i, who competes with $n-1$ other firms in a downstream market. In such models of contracting with externalities —see Segal (1999) and references therein— an agent's valuation may be increasing in a_i and decreasing in a_j, for all $j \neq i$.

Here we show how such settings can be encompassed in our framework. To ease the exposition, for the remainder of the subsection we assume that the allocation set is $A = A_1 \times A_2$, where both A_1 and A_2 are subsets of the real line. Moreover, we assume that A_1 is either finite or a compact interval, and A_2 is finite.[10] An alternative a is an ordered pair $a = (a_1, a_2)$, where $a_1 \in A_1$ and $a_2 \in A_2$. A valuation $w\colon A_1 \times A_2 \to \mathbb{R}$ exhibits *positive externalities* on A_1 and *negative externalities* on A_2 if for all $a_2 \in A_2$, $w(\cdot, a_2)$ is strictly increasing on A_1 and continuous whenever A_1 is a compact interval, and for all $a_1 \in A_1$, $w(a_1, \cdot)$ is strictly decreasing on A_2.

In licensing models of technological innovations by an upstream monopolist to n firms competing in a downstream oligopoly —e.g., Katz and Shapiro (1986)— $A_1 = \{0, 1\}$ represents the access of a firm to the new technology. If all firms are approximately identical, then profits depend on the number, not the identity, of competitors who share license to the innovation. Thus, it is natural to let $A_2 = \{0, 1, 2, \ldots, n-1\}$, assuming that $w(a_1, a_2)$ is increasing in a_1 and decreasing in a_2. This setting also covers classic takeover models with "atomistic" stockholders —e.g., Grossman and Hart (1980), Burkart et al. (1998)— where $w(a_1, a_2)$ represents

[10] Additional work is required to accommodate the more general case where A_2 is countably infinite, but there seems to be few additional economic applications covered under such assumption.

the expected value of the firm's shares held by an agent who tenders $a_2 \in A_2 = \{0, 1, \ldots, k\}$ of his shares to a corporate raider when all other stockholders are tendering a proportion $a_1 \in [0, 1]$ of the firm's total shares. With a superior raider, w is increasing in a_1 and decreasing in a_2.

Corollary 7. *Let $A = A_1 \times A_2$, where $A_1 \subset \mathbb{R}$ is either finite or a compact interval, and $A_2 \subset \mathbb{R}$ is finite. Let $D_7 \subseteq \mathbb{R}^A$ be the space of all valuations that exhibit positive externalities on A_1 and negative externalities on A_2. Then every 2-cycle monotone allocation rule $f \colon D_7 \to A$ is truthfully implementable and satisfies revenue equivalence.*

Obviously, our last result does not depend on the sign of the externalities. It may be possible that valuations are strictly increasing in both arguments, as in models of diffusion of innovations of technology standards —e.g., Dybvig and Spatt (1983)— where the payoffs of an agent depend positively on the (aggregate) adoption choices of other agents, and so forth.

7. Concluding Remarks

In this paper, we introduced the notion of *monotonic transformations in differences* that a preference domain may (or not) admit in a general mechanism design setting with quasilinear utilities and transfers. These attributes are analogues of Maskin's (1999) monotonic transformations wildly used in the social choice literature (without quasilinear preferences). They are easy to understand and have surprisingly strong consequences. In particular, we demonstrated that every domain that admits MD1 and MD2 transformations is a revenue monotonicity domain. We argued in favor of the usefulness of our new attributes by establishing that a large variety of economic domains admit monotonic transformations, including applications to public good provision and other divisible resource allocation problems, multi-unit auction-like environments and other indivisible resource allocation problems, and situations where allocative externalities are present. In some cases, the allocation set we considered is countably infinite or even uncountable. We have focused on providing sufficient conditions on the domain D to be able to achieve truthful implementability and revenue equivalence from 2-cycle monotonicity. We leave other considerations, such as revenue maximization or interim efficiency, for future work.

The type of monotonic transformations we employed here is related to the notion of domain flexibility that was used by Carbajal et al. (2013) to study Roberts' Theorem. The differences between these concepts are twofold. First, monotonic transformations are required to be taken around one and two alternatives, whereas flexibility demands distortions around one, two, and three allocations. Second, monotonic transformations are performed over individual valuations, thus is a condition imposed over individual domains, while flexibility demands similar distortions for all agents simultaneously. On the other hand, in light of the results reported by Nath and Sen (2014) on Roberts' Theorem and selfish preferences, it may be possible to weaken this last requirement of flexibility when the allocation rule is non-bossy by partitioning the set of alternatives and then applying flexibility on individual preference domains. We leave these explorations for future research.

Appendix A. Omitted Proofs

Proof of Lemma 8. To obtain a contradiction, suppose there is $\langle x, y\rangle$ in A such that $\ell_f(x,y) + \ell_f(y,x) > 0$. Assume without loss of generality that $\ell_f(x,y) > 0$, and let $v^x \in f^{-1}(x)$. Choose a real number δ that satisfies

$$\triangle v^x(x,y) \;\geq\; \ell_f(x,y) \;>\; \delta \;>\; -\ell_f(y,x).$$

Using a PMD* transformation obtains $v \in D$ such that $\triangle v(x,y) = \delta$. Moreover, for all other $a \neq x, y$, we have $\triangle v(x,a) > \triangle v^x(x,a)$. This last inequality and the 2-cycle monotonicity of f allows us to conclude that $v \in f^{-1}(x) \cup f^{-1}(y)$. Since $\ell_f(x,y) > \triangle v(x,y) \;=\; \delta > -\ell_f(y,x)$, a contradiction is obtained. □

Proof of Lemma 9. Let $\{b^1, b^2, \ldots, b^m\}$ be an arbitrary collection of consecutive elements in A, that is $\langle b^j, b^{j+1}\rangle$ for all $j = 1, \ldots, m-1$. From the assumption of the lemma, it is evident that one has

$$\ell_f(b^1, b^m) \;\geq\; \sum\nolimits_{j=1}^{m-1} \ell_f(b^j, b^{j+1}). \tag{10}$$

Let $\{a_1, \ldots, a_k, a_{k+1} = a_1\}$ be an arbitrary cycle in A. Since $\prec$ is a linear order on A, for each $i = 1, \ldots, k-1$, we can find a collection $\{a_i = b_i^1, b_i^2, \ldots, b_i^{m_i} = a_{i+1}\}$ of consecutive pairs. Using Eq. (10) we conclude

$$\ell_f(a_i, a_{i+1}) \;\geq\; \sum\nolimits_{j=1}^{m_i-1} \ell_f(b_i^j, b_i^{j+1}), \qquad \text{for all } i = 1, \ldots, k-1.$$

Notice now that the path

$$\{\, a_1 = b_1^1, \ldots, b_1^{m_1} = a_2 = b_2^1, \ldots, b_2^{m_2} = a_3 = b_3^1, \ldots, b_{k-1}^1, \ldots, b_{k-1}^{m_{k-1}} = a_k \,\}$$

is composed of consecutive pairs and connects a_1 to a_k. But this means the reverse path is also formed of consecutive pairs and connects a_k to $a_1 = a_{k+1}$. Consequently, using Eq. (10) again and the fact that by assumption $\ell_f(x,y) + \ell_f(y,x) = 0$ for all $\langle x, y\rangle$ in A, we obtain

$$\sum\nolimits_{i=1}^{k} \ell_f(a_i, a_{i+1}) \;\geq\; \sum\nolimits_{i=1}^{k-1} \sum\nolimits_{j=1}^{m_i-1} \left[\, \ell_f(b_i^j, b_i^{j+1}) \,+\, \ell_f(b_i^{j+1}, b_i^j) \,\right] \;=\; 0. \qquad \square$$

Proof of Lemma 10. Let $\langle x, y\rangle$, $z \in A$ such that $x \prec y \prec z$ or $z \prec y \prec x$, and $\epsilon > 0$. Using a MD1 transformation yields to a $v^x \in f^{-1}(x)$ such that

$$\ell_f(x,z) \;\leq\; \triangle v^x(x,z) \;<\; \ell_f(x,z) + \epsilon \quad \text{ and } \quad \ell_f(x,y) \;<\; \triangle v^x(x,y), \tag{11}$$

where $\ell_f(x,y) \;=\; -\ell_f(y,x)$ because PMD is present and x, y are consecutive elements. Without loss of generality, we can assume that $\ell_f(x,y) \geq 0$.

Now choose $0 < \epsilon' < \epsilon$ sufficiently small so that $\delta = \ell_f(x,y) - \epsilon' > -\epsilon$. Using a PMD transformation obtains a valuation $v \in D$ such that $\triangle v(x,y) = \delta$ and for every other $a \in A \setminus \{x, y\}$, $\triangle v(x,a) > \triangle v^x(x,a)$. Using previous arguments, this implies $f(v) = y$. Moreover, v can be chosen to satisfy $\triangle v(x,z) < \triangle v^x(x,z) + \epsilon'$. Consequently,

$$\ell_f(y,z) \;\leq\; \triangle v(y,z) \;=\; \triangle v(y,x) \,+\, \triangle v(x,z) \;<\; -\ell_f(x,y) \,+\, \triangle v^x(x,z) \,+\, 2\epsilon'.$$

Combining this last expression with the first part of expression (11) yields to

$$\ell_f(x,y) \,+\, \ell_f(y,z) \;<\; \ell_f(x,z) \,+\, \epsilon \,+\, 2\epsilon'.$$

The preceding argument is valid for arbitrarily small $0 < \epsilon' < \epsilon$. Henceforth, we conclude that for $\langle x, y\rangle$ and $z \in A$ such that $x \prec y \prec z$ or $z \prec y \prec x$, one has $\ell_f(x, y) + \ell_f(y, z) \leq \ell_f(x, z)$, as desired. □

Proof of Corollary 1. By Theorem 1, it suffices to argue that D_1 admits MD1 and MD2 transformations. To show MD1, fix $w \in D_1$, $x \in A$ and $\epsilon > 0$. For simplicity, assume that $\underline{a} < x < \overline{a}$. The proof for the end-point cases can be adapted from the arguments below. Let $dw(x-)$ and $dw(x+)$ denote the left and right Dini derivatives of w at x, respectively:

$$\begin{aligned} dw(x-) &= \liminf_{h\to 0-} \tfrac{w(x+h) - w(x)}{h}, \\ dw(x+) &= \limsup_{h\to 0+} \tfrac{w(x+h) - w(x)}{h}. \end{aligned}$$

Consider an affine function ϕ_1 with slope equal to $2dw(x-)$ passing through x, and a second affine function ϕ_2 with slope $(1/2)dw(x+)$ passing through x. Choose alternative $b_1 < x$ to be sufficiently close to x for the following to be satisfied, for all $b_1 < a < x$:

$$0 \;<\; w(a) - \phi_1(a) \;<\; w(b_1) - \phi_1(b_1) \;=\; \xi_1 \;<\; \epsilon.$$

Similarly, choose $x < b_2$ close enough to x so that for all $x < a < b_2$:

$$0 \;<\; w(a) - \phi_2(a) \;<\; w(b_2) - \phi_2(b_2) \;=\; \xi_2 \;<\; \epsilon.$$

Finally, construct a function $\tilde{v}$ defined on A as follows:

$$\tilde{v}(a) \;=\; \begin{cases} w(a) - \xi_1 & : \; \underline{a} \leq a < b_1, \\ \phi_1(a) & : \; b_1 \leq a \leq x, \\ \phi_2(a) & : \; x \leq a \leq b_2, \\ w(a) - \xi_2 & : \; b_2 < a \leq \overline{a}. \end{cases}$$

Note that $\tilde{v}$ is continuous, strictly increasing, coincides with w only at x and is everywhere else strictly below w. Readily, for all $a \in A$, $a \neq x$, one has

$$\epsilon \;>\; \triangle\tilde{v}(x, a) \;-\; \triangle w(x, a) \;>\; 0.$$

Because we only care about valuation differences, for $\kappa > 0$ sufficiently large, $v = \tilde{v} + \kappa$ belongs to D_1. It follows that MD1 is in place.

We show that D_1 admits MD2 transformations. Let $x, y \in A$, $x \neq y$, $v^x, v^y \in D_1$, $\epsilon > 0$, and $\delta \in \mathbb{R}$ satisfy $\triangle v^x(x, y) > \delta > \triangle v^y(x, y)$. Because D_1 is closed under positive shifts, using the arguments in Lemma 2 it is without loss of generality that we assume $v^x(x) > v^y(x)$, $v^y(y) > v^x(y)$ and $v^x(x) - v^y(y) = \delta$. From the arguments used in MD1, we deduce the existence of strictly increasing, continuous functions $\tilde{v}^x$ and $\tilde{v}^y$, such that $\tilde{v}^x \leq v^x$ with strict inequality everywhere except at x, $\tilde{v}^y \leq v^y$ with strict inequality everywhere except at y, and further

$$\epsilon \;>\; \triangle\tilde{v}^x(x, a) \;-\; \triangle v^x(x, a) \;>\; 0, \quad \text{for all } a \neq x, \tag{12}$$

$$\epsilon \;>\; \triangle\tilde{v}^y(y, a) \;-\; \triangle v^y(y, a) \;>\; 0, \quad \text{for all } a \neq y. \tag{13}$$

Now let $\tilde{v} = \tilde{v}^x \vee \tilde{v}^y$, noticing that $\tilde{v}(x) = \tilde{v}^x(x) = v^x(x)$ and $\tilde{v}(y) = \tilde{v}^y(y) = v^y(y)$. For sufficiently large $\kappa > 0$, it will follow that $v = \tilde{v} + \kappa$ belongs to D_1. We obtain $\triangle v(x, y) = \triangle\tilde{v}(x, y) = \delta$. Further, for all $a \neq x, y$ such that $\tilde{v}^x(a) \geq \tilde{v}^y(a)$ one has

$$\triangle v(x, a) \;-\; \triangle v^x(x, a) \;=\; \triangle\tilde{v}^x(x, a) \;-\; \triangle v^x(x, a)$$

whereas for all $a \neq x, y$ such that $\tilde{v}^x(a) < \tilde{v}^y(a)$ one has

$$\triangle v(y,a) \;-\; \triangle v^y(x,a) \;=\; \triangle \tilde{v}^y(y,a) \;-\; \triangle v^y(y,a).$$

In this case, additionally we obtain

$$\triangle v(x,a) \;-\; \triangle v^x(x,a) \;=\; v^x(a) \;-\; \tilde{v}^y(a) \;\leq\; \tilde{v}^x(a) \;-\; \tilde{v}^y(a) \;<\; 0.$$

In light of Eq. (12) and (13), the three last expressions give us the result. □

Proof of Corollary 2. By assumption, there exits a contraction (D_1, g) of the allocation problem (D_2, f), where $g\colon D_1 \to A$ is surjective and satisfies $g = f|_{D_1}$. Since f is 2-cycle monotone, so is g. Since D_1 admits MD1 and MD2 transformations, for all $x, y, z \in A$, one has $\ell_g(x,y) + \ell_g(y,z) + \ell_g(z,x) = 0$. Using Lemma 7 we conclude that $\ell_f(a,b) = \ell_g(a,b)$ for all $a, b \in A$, and therefore $f\colon D_2 \to A$ is also truthfully implementable and satisfies revenue equivalence. □

Proof of Corollary 3. We begin with the following observation —for details, see Hirsch (1976). For any $x \in A$ and open neighborhood B of x, there exists a $\mathscr{C}^r$ function μ mapping A to $[0,1]$ that takes values $\mu(x) = 1$, $0 < \mu(b) < 1$ for all $b \in B \setminus \{x\}$, and $\mu(a) = 0$ for all $a \in A \setminus B$.

To obtain MD1, fix $x \in A$, $w \in D_3$ and $\epsilon > 0$. Choose $\xi \in \mathbb{R}$ such that $0 < \xi < \epsilon$. Using our previous observation, let B be a neighborhood of x and λ be a $\mathscr{C}^r$ function satisfying $\lambda(x) = 0$, $0 < \lambda(b) < 1$ for all $b \in B \setminus \{x\}$, and $\lambda(a) = 1$ for all $a \in A \setminus B$. Now consider the function v defined on A by $v = w - \xi\,\lambda$. Clearly, v belongs to D_3. Moreover, for all $a \in A$, $a \neq x$, it follows that

$$0 \;<\; \triangle v(x,a) \;-\; \triangle w(x,a) \;=\; \xi\,\lambda(a) \;<\; \epsilon.$$

To show MD2, fix distinct alternatives $x, y \in A$, valuations $v^x, v^y \in D_3$, $\epsilon > 0$ and $\delta \in \mathbb{R}$ satisfying $\triangle v^x(x,y) > \delta > \triangle v^y(x,y)$. Because D_3 is closed under positive shifts, we assume without loss of generality that $v^x(x) > v^y(x)$, $v^y(y) > v^x(y)$ and $v^x(x) - v^y(y) = \delta$. Let B^x and B^y be open neighborhoods of x and y, respectively, with disjoint closure, such that $v^x(b) > v^y(b)$ for all $b \in B^x$ and $v^y(b) > v^x(b)$ for all $b \in B^y$. Following our preliminary observation, there exist functions $\lambda^x, \lambda^y \in D_3$ such that $\lambda^x(x) = 0$, $0 < \lambda^x(b) < 1$ for all $b \in B^x \setminus \{x\}$, and $\lambda^x(a) = 1$ for all $a \in A \setminus B^x$, and similarly $\lambda^y(y) = 0$, $0 < \lambda^y(a) < 1$ for all $a \in B^y \setminus \{y\}$, and $\lambda^y(a) = 0$ for all $a \in A \setminus B^y$.

With these elements, construct valuations $\tilde{v}^x = v^x - \xi\,\lambda^x$ and $\tilde{v}^y = v^y - \xi\,\lambda^y$, for a real number $0 < \xi < \epsilon$, and define the function

$$v \;=\; \tilde{v}^x \vee \tilde{v}^y.$$

Clearly, $v \in D_3$. One can easily verify that this is our desired valuation using the observations at the end of the proof of Corollary 1. □

Proof of Corollary 4. The arguments exhibited in Lemma 2 can be adapted to establish that the domain of non-negative, strictly increasing functions on $A = \{a_1, a_2, \ldots\}$ admits MD1 and MD2 transformations. Apply now Lemma 7. □

Proof of Corollary 5. By Theorem 2, it suffices to show that D_5 admits MD1 and PMD transformations. We first point out that since D_5 is locally open, MD1 is readily obtained. To show that PMD is satisfied, fix $\langle x, y\rangle$ in A, and assume without loss of generality that x is an immediate successor of y. Let $w \in D$ for which $\triangle w(x,y) \geq 0$, $\epsilon > 0$, and $\delta \in \mathbb{R}$ such that $\triangle w(x,y) > \delta > -\epsilon$. Choose real numbers $0 < \xi^x < \epsilon$ and $0 < \xi^y$ that satisfy $\xi^x - \xi^y = \delta - \triangle w(x,y)$. With these numbers, construct a function v on A as follows: for any $a \prec y$, let $v(a) = w(a)$; $v(y) = w(y) + \xi^y$ and $v(x) = w(x) + \xi^x$; for any $a \succ x$, choose $v(a) < w(a)$ in a way that the resulting valuation v is single-peaked (for instance, setting $v(a) < \min\{w(x), w(a)\}$ and strictly decreasing will do). It can be readily verified that $v \in D_5$ is our desired valuation. This suffices for PMD to be present. □

Proof of Corollary 6. It suffices to show that D_6 admits MD1 and PMD transformations, which can be accomplished along the lines of the proof of Corollary 5 above. We omit the details. □

Proof of Corollary 7. By Theorem 1, it suffices to show that D_7 admits MD1 and MD2 transformations. Assume that $A_1 \subset \mathbb{R}$ is a compact interval $A_2 \subset \mathbb{R}$ is finite —the case where A_1 is finite is readily adapted from our arguments below. We begin with the following observation. Fix $w \in D_7$ and $a_2, b_2 \in A_2$ such that $a_2 < b_2$. Then the function $a_1 \mapsto w(a_1, a_2) - w(a_1, b_2)$ is strictly positive and continuous on A_1, and as a consequence of the compactness of A_1, its minimum $\phi_w(a_2, b_2) = \min\{w(a_1, a_2) - w(a_1, b_2) \colon a_1 \in A_1\}$ is also strictly positive. Since A_2 is finite, we obtain that

$$\min\{\phi_w(a_2, b_2) \colon a_2, b_2 \in A_2,\ a_2 < b_2\} \;\equiv\; \phi_w \;>\; 0. \tag{14}$$

With an alternative $\hat{a}_2 \in A_2$ fixed, one has that for any $\xi \in \mathbb{R}$ such that $|\xi| < \phi_w$, the function $\tilde{w}$ defined on A by

$$\tilde{w}(a) \;=\; \begin{cases} w(a_1, \hat{a}_2) & : \text{ for all } a_1 \in A_1, \text{ for } \hat{a}_2 \in A_2, \\ w(a_1, a_2) + \xi & : \text{ for all } a_1 \in A_1, \text{ all } a_2 \in A_2 \setminus \{\hat{a}_2\}, \end{cases}$$

is strictly increasing and continuous on A_1 and strictly decreasing on A_2, thus it belongs to D_7.

To show MD1, fix alternative $x = (x_1, x_2) \in A_1 \times A_2$, valuation $w \in D$ and $\epsilon > 0$. Choose $0 < \xi < \min\{\epsilon, \phi_w\}$, for ϕ_w as defined in Eq. (14). The function $w(\cdot, x_2)$ is strictly increasing and continuous on A_1. Therefore, the argument employed in the proof of Corollary 1 provides us with a strictly increasing, continuous function $v(\cdot, x_2)$ defined on A_1 satisfying $v(x_1, x_2) = w(x_1, x_2)$ and further

$$0 \;<\; \triangle v((x_1, x_2), (a_1, x_2)) \;-\; \triangle w((x_1, x_2), (a_1, x_2)) \;<\; \xi,$$

for all $a_1 \in A_1$, $a_1 \neq x_1$. For any other $a_2 \neq x_2$ in A_2, define the function $v(\cdot, a_2)$ on A_1 by $v(a_1, a_2) = w(a_1, a_2) - \xi$. Put together, it follows that v defined on $A_1 \times A_2$ belongs to D_7. Moreover, for any $a = (a_1, a_2)$ with $a_2 \neq x_2$, we have that

$$\triangle v((x_1, x_2), (a_1, a_2)) \;-\; \triangle w((x_1, x_2), (a_1, a_2)) \;=\; \xi.$$

Therefore, for every alternative $a \neq x$ in A, one has $0 < \triangle v(x,a) - \triangle w(x,a) < \epsilon$, as desired.

To show MD2, fix alternatives $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in A, with $x \neq y$, valuations v^x, v^y in D_7, and $\delta \in \mathbb{R}$ such that $\triangle v^x(x, y) > \delta > \triangle v^y(x, y)$. Additionally, fix an arbitrary $\epsilon > 0$ and let

$$0 \;<\; \xi \;<\; \min\{\epsilon, \phi_{v^x}, \phi_{v^y}\},$$

for ϕ_{v^x} and ϕ_{v^y} as in Eq. (14), replacing w with v^x and v^y, respectively. From previous arguments, it is without loss of generality that we assume v^x, v^y satisfy $v^x(x) - v^y(y) = \delta$. Notice that this implies

$$v^x(x_1, x_2) \;>\; v^y(x_1, x_2) \quad \text{and} \quad v^y(y_1, y_2) \;>\; v^x(y_1, y_2).$$

From MD1, we obtain that there exist functions $\tilde{v}^x(\cdot, x_2), \tilde{v}^y(\cdot, y_2)$ defined on A_1, strictly increasing, continuous and such that $\tilde{v}^x(x_1, x_2) = v^x(x_1, x_2)$, $\tilde{v}^y(y_1, y_2) = v^y(y_1, y_2)$, and further satisfying

$$\begin{aligned}
0 \;&<\; \triangle\tilde{v}^x((x_1, x_2), (a_1, x_2)) \;-\; \triangle v^x((x_1, x_2), (a_1, x_2)) \;<\; \xi, \quad \text{all } a_1 \neq x_1,\\
0 \;&<\; \triangle\tilde{v}^y((y_1, y_2), (a_1, y_2)) \;-\; \triangle v^y((y_1, y_2), (a_1, y_2)) \;<\; \xi, \quad \text{all } a_1 \neq y_1.
\end{aligned}$$

Define now $\tilde{v}^x(\cdot, a_2) = v^x(\cdot, a_2) - \xi$ on A_1, for all $a_2 \neq x_2$, and similarly $\tilde{v}^y(\cdot, a_2) = v^y(\cdot, a_2) - \xi$ on A_1, for all $a_2 \neq y_2$. Finally, we define v on A as

$$v(\,\cdot\,, a_2) \;=\; \tilde{v}^x(\,\cdot\,, a_2) \vee \tilde{v}^y(\,\cdot\,, a_2), \quad \text{for all } a_2 \in A_2.$$

It is readily verified that v belongs to D_7 and is our desired valuation. □

Appendix B. Counter Examples

Example 4 (continued). We verify here that D_{IV} satisfies MD2*. Similar arguments can be used to show MD1 as well. Fix $x, y \in A$. We exhaust all possible cases.

Case 1. Suppose that $v^x = (14 + \alpha^x_x, 8, 2 + \alpha^x_z)$ and $v^y = (14 + \alpha^y_x, 8, 2 + \alpha^y_z)$ belong to D_α. One has that the relation

$$\triangle v^x(x, y) \;=\; 6 + \alpha^x_x \;>\; 6 + \alpha^y_x \;=\; \triangle v^y(x, y)$$

holds if and only if $\alpha^x_x > \alpha^y_x$. Let it be the case and let $\delta \in \mathbb{R}$ satisfy $\triangle v^x(x, y) > \delta > \triangle v^y(x, y)$. Now choose $\xi = 6 + \alpha^x_x - \delta$, noticing that $0 < \xi < \alpha^x_x$. Furthermore, select a real number $\xi' > 0$ small enough to satisfy $0 < \alpha^x_z - \alpha^x_x - \xi' < 1$. With these elements, define a valuation

$$v \;=\; v^x \;-\; \xi\, \mathbb{1}_{\{x,z\}} \;-\; \xi'\, \mathbb{1}_{\{z\}},$$

which belongs to D_α. Moreover, v is constructed so that $\triangle v(x, y) = \triangle v^x(x, y) - \xi = \delta$ and $\triangle v(x, z) - \triangle v^x(x, z) = \xi' > 0$ are both satisfied.

Case 2. Let $v^x = (14 + \alpha^x_x, 8, 2 + \alpha^x_z) \in D_\alpha$ and $v^y = (15 + \beta^y_x, 10 + \beta^y_y, 6) \in D_\beta$. The relation

$$\triangle v^x(x, y) \;=\; 6 + \alpha^x_x \;>\; 5 + \beta^y_x - \beta^y_y \;=\; \triangle v^y(x, y)$$

holds for all permissible parameter values. Let δ be such that $\triangle v^x(x, y) > \delta > \triangle v^y(x, y)$. When $\delta > 6$, proceed as in Case 1. When $\delta < 6$ instead, select $\xi = \delta - 5 - \beta^y_x + \beta^y_y$, noticing $0 < \xi < 1$. Now choose $\xi', \xi'' > 0$ to satisfy $\xi' - \xi'' = \xi$, letting ξ'' be small enough so that $0 < \beta^y_y + \xi'' < 1$. Notice that by construction we have $(\beta^y_x + \xi') - (\beta^y_y + \xi'') = \delta - 5 > 0$ and also $(1 + \beta^y_y + \xi'') - (\beta^y_x + \xi') = 6 - \delta > 0$.

It follows that $0 < \beta_y^y + \xi'' < \beta_x^y + \xi' < 1 + \beta_y^y + \xi''$. With these choices, we now define a valuation

$$v \;=\; v^y \;+\; \xi'\,\mathbb{1}_{\{x\}} \;+\; \xi''\mathbb{1}_{\{y\}},$$

concluding from previous analysis that $v \in D_\beta$. Also, $\triangle v(x,y) = \triangle v^y(x,y) + \xi = \delta$ and $\triangle v(y,z) - \triangle v^y(y,z) = \xi'' > 0$, as desired.

Case 3. Suppose that $v^x = (14 + \alpha_x^x, 8, 2 + \alpha_z^x)$ belongs to D_α and the valuation $v^y = (10, 2 + \gamma_y^y, \gamma_z^y - 1)$ to D_γ. It follows that the relation

$$\triangle v^x(x,y) \;=\; 6 + \alpha_x^x \;>\; 8 - \gamma_y^y \;=\; \triangle v^y(x,y)$$

holds if and only if $2 > \gamma_y^y > 2 - \alpha_x^x$, for $0 < \alpha_x^x < 1$. For a number δ such that $\triangle v^x(x,y) > \delta > \triangle v^y(x,y)$, proceed as in Case 1 to obtain the desired valuation.

Case 4. When $v^x = (15 + \beta_x^x, 10 + \beta_y^x, 6) \in D_\beta$ and $v^y = (14 + \alpha_x^y, 8, 2 + \alpha_z^y) \in D_\alpha$, we obtain the relation

$$\triangle v^x(x,y) \;=\; 5 + \beta_x^x - \beta_y^x \;>\; 6 + \alpha_x^y \;=\; \triangle v^y(x,y),$$

which never holds for permissible values of β_x^x, β_y^x and α_x^y.

Case 5. Suppose that $v^x = (15 + \beta_x^x, 10 + \beta_y^x, 6)$ and $v^y = (15 + \beta_x^y, 10 + \beta_y^y, 6)$ are both in D_β. The inequality

$$\triangle v^x(x,y) \;=\; 5 + \beta_x^x - \beta_y^x \;>\; 5 + \beta_x^y - \beta_y^y \;=\; \triangle v^y(x,y)$$

now holds if and only if $0 < \beta_x^y - \beta_y^y < \beta_x^x - \beta_y^x < 1$. For a number δ such that $\triangle v^x(x,y) > \delta > \triangle v^y(x,y)$, proceed as in Case 2 to obtain the desired valuation.

Case 6. When $v^x = (15 + \beta_x^x, 10 + \beta_y^x, 6) \in D_\beta$ and $v^y = (10, 2 + \gamma_y^y, \gamma_z^y - 1) \in D_\gamma$, the inequality

$$\triangle v^x(x,y) \;=\; 5 + \beta_x^x - \beta_y^x \;>\; 8 - \gamma_y^y \;=\; \triangle v^y(x,y)$$

never holds for admissible values of β_x^x, β_y^x and γ_y^y.

Case 7. Let $v^x = (10, 2 + \gamma_y^x, \gamma_z^x - 1) \in D_\gamma$ and $v^y = (14 + \alpha_x^y, 8, 2 + \alpha_z^y) \in D_\alpha$. Then we have that

$$\triangle v^x(x,y) \;=\; 8 - \gamma_y^x \;>\; 6 + \alpha_x^y \;=\; \triangle v^y(x,y),$$

which holds for all parameter values of α_x^y and γ_y^x for which $2 > \alpha_x^y + \gamma_y^x$. Now let δ be a real number such that $\triangle v^x(x,y) > \delta > \triangle v^y(x,y)$. When $\delta > 7$, choose $\xi = 8 - \gamma_y^x - \delta > 0$, noticing that $0 < \gamma_z^x < \gamma_y^x < \gamma_y^x + \xi < 1 + \gamma_z^x$, where the last inequality follows because $(1 + \gamma_z^x) - (\gamma_y^x + \xi) = \delta - 7 + \gamma_z^x > 0$. Define then the valuation

$$v \;=\; v^x \;+\; \xi\,\mathbb{1}_{\{y\}} \;-\; \xi'\mathbb{1}_{\{z\}}.$$

For sufficiently small $\xi' > 0$, this valuation belongs to D_γ. Moreover, $\triangle v(x,y) = \triangle v^x(x,y) - \xi = \delta$, and $\triangle v(x,z) - \triangle v^x(x,z) = \xi' > 0$.

When we have $\delta < 7$ instead, choose $\xi = \delta - 6 - \alpha_x^y$, noticing that we obtain $0 < \alpha_x^y + \xi < 1$. Select $\xi' > 0$ to be arbitrarily close to zero, and define

$$v \;=\; v^y \;+\; \xi\,\mathbb{1}_{\{x\}} \;-\; \xi'\mathbb{1}_{\{z\}}.$$

By construction, v belongs in this case to D_α and satisfies $\triangle v(x,y) = \triangle v^y(x,y) + \xi = \delta$, and $\triangle v(y,z) - \triangle v^y(y,z) = \xi' > 0$, as desired.

Case 8. Let $v^x = (10, 2 + \gamma_y^x, \gamma_z^x) \in D_\gamma$ and $v^y = (15 + \beta_x^y, 10 + \beta_y^y, 6) \in D_\beta$. The expression

$$\triangle v^x(x,y) \;=\; 8 - \gamma_y^x \;>\; 5 + \beta_x^y - \beta_y^y \;=\; \triangle v^y(x,y),$$

holds for all admissible values of the parameters γ_y^x, β_x^y and β_y^y. Let $\triangle v^x(x,y) > \delta > \triangle v^y(x,y)$. When $\delta > 7$, choose $\xi = 8 - \gamma_y^x - \delta$ and proceed as in the first part of Case 7, obtaining v from a transformation of v^x. When $\delta < 6$ instead, choose $\xi = \delta - 5 - \beta_x^y + \beta_y^y$ and proceed as in the second part of Case 2, transforming v^y to obtain v.

When $6 < \delta < 7$, for some parameter values it may not be possible to distort v^x to reach an admissible valuation v whose value difference at x and y equal to δ, and for which $\triangle v(x,z) - \triangle v^x(x,z) > 0$. This happens, in particular, when $\gamma_z^x < 7 - \delta$. In this case, let $\alpha_x = \delta - 6$, noticing $0 < \alpha_x < 1$, and select further a real number α_z such that $\alpha_x < \alpha_z < 1 + \alpha_x$. With these parameter values, define

$$v \;=\; (14 + \alpha_x, 8, 2 + \alpha_z) \tag{15}$$

which belongs to D_α. Readily, $\triangle v(x,y) = 6 + \alpha_x = \delta$, and

$$\triangle v(x,z) \;-\; \triangle v^x(x,z) \;=\; (1 + \alpha_x - \alpha_z) \;+\; \gamma_z^x \;>\; 0,$$

where the last inequality follows from the fact that the expression in parentheses is positive, as $v \in D_\alpha$.

Case 9. Suppose that $v^x = (10, 2 + \gamma_y^x, \gamma_z^x - 1)$ and $v^y = (10, 2 + \gamma_y^y, \gamma_z^y - 1)$ belong to D_γ. We see that the relation

$$\triangle v^x(x,y) \;=\; 8 - \gamma_y^x \;>\; 8 - \gamma_y^y \;=\; \triangle v^y(x,y)$$

holds if and only if $2 > \gamma_y^y > \gamma_y^x > 0$. Let it be the case and let $\delta \in \mathbb{R}$ be such that $\triangle v^x(x,y) > \delta > \triangle v^y(x,y)$. If $7 < \delta < 8$, then we can generate an admissible v meeting all required properties by transforming v^x as in the first part of Case 8. If $6 < \delta < 7$, then as in the last part of Case 8 we construct a valuation $v \in D_\alpha$ for which all the required properties hold.

Cases 1 to 9 above show that MD2* is satisfied for $x, y \in A$. A similar analysis shows that MD2* is satisfied for x, z and y, z. Thus, the domain $D_{IV} = D_\alpha \cup D_\beta \cup D_\gamma$ admits MD2* transformations. However, MD2 is not present. In the last part of Case 8, the only way to obtain a valuation that satisfies MD2* is by putting $v \in D_\alpha$ such that $\triangle v(x,z) - \triangle v^x(x,z) = 1 + \alpha_x - \alpha_z + \gamma_y^x$, which is a difference bounded below by $\gamma_y^x > 0$. $\diamond$

Example 5. This is adapted from Heydenreich et al. (2009) to show that the sufficient condition of Lemma 1 is *not* necessary for revenue equivalence when the allocation set is infinite. Let $A = [0,1]$ be the allocation set. For each $x \in A$, define the function $v^x \colon A \to \mathbb{R}$ by

$$v^x(a) \;=\; \begin{cases} 0 & : \; a \geq x, \\ a - x & : \; a < x. \end{cases}$$

A common interpretation is that $x \in A$ represents the minimal amount of a divisible good demanded by the agent when endowed with valuation v^x. The agent experiences zero disutility when his demand is met (i.e., when the chosen alternative $a \geq x$), otherwise he experiences a linear disutility. Let $D_V = \{v^x : \; x \in A\}$ be the allocation domain.

Consider $f \colon D_V \to A$ defined by $f(v^a) = a$, for all $a \in A$. It is well known that this particular allocation rule is implementable but does not satisfy revenue equivalence.[11] Therefore, it will be impossible to find a directed path between any two distinct alternatives a, b, for which the f-length of every consecutive 2-cycle is equal to zero. To see this, note for every $x \in A$, $f^{-1}(x) = \{v^x\}$. Also, for all $a, b \in A$, $a \neq b$, one has $\ell_f(a,b) + \ell_f(b,a) = |a-b| > 0$.

Consider instead the allocation rule $g\colon V \to A$ defined by $g(v^a) = a/2$, for all $a \in A$.[12] In this case, of course, $g^{-1}(a/2) = \{v^a\}$. It is also well understood that g is implementable and satisfies the revenue equivalence property. We want to show now that there exists no positive integer K that bounds the number of nodes needed to connect pairs of alternatives in the allocation set. Let $x, y \in A$, $x \neq y$, be such that $y/2 \leq x \leq 2y$. In this case one immediately observes that:

$$-\ell_g\big(\tfrac{y}{2}, \tfrac{x}{2}\big) \;=\; (x-y)/2 \;=\; \ell_g\big(\tfrac{x}{2}, \tfrac{y}{2}\big).$$

Let now $x, y \in A$, $x \neq y$ be such that $x < y/2$. One has:

$$-\ell_g\big(\tfrac{y}{2}, \tfrac{x}{2}\big) \;=\; (x-y)/2 \;<\; -x/2 \;=\; \ell_g\big(\tfrac{x}{2}, \tfrac{y}{2}\big).$$

One can find a finite path $\{x/2 = a_0, a_1, \ldots, a_k = y/2\}$ such that for each $i = 0, 1, \ldots, k-1$, $a_i/2 \leq a_{i+1} \leq 2a_i$ is satisfied. However, there is no positive integer $K < \infty$ that uniformly bounds the cardinality of the paths that are used to connect all pairs of alternatives in A. For instance, let $y = 1$ and $x = 1/m$, where m is a positive integer. One can show without much difficulty that the number of nodes k needed to connect $x/2$ to $y/2$ is dependent on m by the following relationship: $m \leq 2^{k-1}$. ◇

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[11]The earliest reference to this type of examples that we are aware of is Holmström (1979).

[12]Here the allocation rule under consideration is not surjective, but this is inconsequential for the point the example is illustrating.

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